REGULARITY OF SOLUTIONS TO CHARACTERISTIC INITIAL-BOUNDARY VALUE PROBLEMS FOR SYMMETRIZABLE SYSTEMS

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Abstract. We consider the initial-boundary value problem for linear Friedrichs symmetrizable systems with characteristic boundary of constant rank. We assume the existence of the strong L^2 solution satisfying a suitable energy estimate, but we do not assume any structural assumption sufficient for existence, such as the fact that the boundary conditions are maximally dissipative or the Kreiss-Lopatinski condition. We show that this is enough in order to get the regularity of solutions, in the natural framework of weighted anisotropic Sobolev spaces, provided the data are sufficiently smooth.

1. Introduction and main results

It is well-known that for solutions of symmetric hyperbolic systems with characteristic boundary the full regularity (i.e. solvability in the usual Sobolev spaces H^m) cannot be expected generally because of the possible loss of derivatives in the normal direction to the boundary, see [42, 18].

The natural space is the anisotropic Sobolev space H^m_*, which comes from the observation that the one order gain of normal differentiation should be compensated by two order loss of tangential differentiation; such a fact was first found in [7]. The theory has been developed mostly for characteristic boundaries of constant multiplicity (see the definition in assumption (B)) and maximally nonnegative boundary conditions, see [7, 12, 20, 27, 29, 30, 31, 37]. For more facts about the H^m_* spaces we also refer to [19, 34, 38] and to Appendix B at the end of this paper. Function spaces of this type have also been considered in [1, 10].

Even if the boundary is characteristic, it is not always needed to make use of H^m_* spaces. An important example is that of Euler equations, where, thanks to the vorticity equation, one can find the solution in the usual Sobolev spaces H^m, see [3, 26, 9].

The equations of ideal Magneto-hydrodynamics provide an important example of ill-posedness in Sobolev spaces H^m, see [18]. Application to MHD of H^m_* spaces may be found in [43, 28, 35]. Applications to general relativity are in [11, 39], see also [25]. An extension to nonhomogeneous strictly dissipative boundary conditions has been considered in [5, 36]. For problems with a nonuniformly characteristic boundary we refer to [17, 23, 32, 33].

There are important characteristic problems of physical interest where boundary conditions are not maximally nonnegative. Under the more general Kreiss-Lopatinski condition (KL), the theory has been developed for problems satisfying the *uniform* KL condition with uniformly characteristic boundaries (when the boundary matrix has constant rank in a neighborhood of the boundary), see [16, 4] and references therein.

However, these assumptions seem to be too restrictive for many problems of physics. For example, in case of vortex-sheets for compressible Euler equations the KL condition holds only in weak form [8, 9]. For current-vortex sheets in ideal MHD, there also holds a weak KL condition and the free boundary is characteristic, but not uniformly characteristic [41].

We think that it can be convenient to distinguish between the L^2 existence theory of weak solutions, and the regularity theory. In the present paper we are interested in the latter question. We assume the existence of the strong L^2 solution satisfying a suitable energy estimate, without assuming any structural assumption sufficient for existence, such as the fact that the boundary conditions are maximally dissipative

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or satisfy the Kreiss-Lopatinski condition. We show that this is enough in order to get the regularity of solutions, in the natural framework of weighted anisotropic Sobolev spaces H^m_*, provided the data are sufficiently smooth. Obviously, the present results contain in particular what has been previously obtained for maximally nonnegative boundary conditions.

For a given integer $n \geq 2$, let Ω be an open bounded subset of $\mathbb{R}^n$, lying locally on one side of its smooth connected boundary $\partial\Omega$. For $T > 0$ we set $Q_T = \Omega \times]0, T[$ and $\Sigma_T = \partial\Omega \times]0, T[$. For $T = +\infty$ we set $Q_\infty = \Omega \times]0, +\infty[$ and $\Sigma_\infty = \partial\Omega \times]0, +\infty[$. We are interested in the following initial-boundary value problem (shortly written IBVP)

$$Lu = F, \qquad \text{in } Q_T \tag{1}$$

$$Mu = G, \qquad \text{on } \Sigma_T \tag{2}$$

$$u_{|t=0} = f, \qquad \text{in } \Omega, \tag{3}$$

where L is a first order linear partial differential operator

$$L = \partial_t + \sum_{i=1}^n A_i(x,t)\partial_i + B(x,t), \tag{4}$$

$\partial_t := \frac{\partial}{\partial t}$ and $\partial_i := \frac{\partial}{\partial x_i}$, $i = 1, \dots, n$.

The coefficients A_i, B, for $i = 1, \dots, n$, are real $N \times N$ matrix-valued functions, defined on Q_∞. The unknown $u = u(x,t)$, and the data $F = F(x,t)$, $G = G(x,t)$, $f = f(x)$ are vector-valued functions with N components, defined on $\overline{Q}_T$, $\overline{\Sigma}_T$ and $\overline{\Omega}$ respectively. $M = M(x,t)$ is a given real $d \times N$ matrix-valued function; M is supposed to have maximal constant rank d, everywhere on Σ_∞.

Let $\nu(x) = (\nu_1(x), \dots, \nu_n(x))$ be the unit outward normal to $\partial\Omega$ at a point x; then

$$A_\nu(x,t) = \sum_{i=1}^n A_i(x,t)\nu_i(x)$$

is called the *boundary matrix*. Let $P(x,t)$ be the orthogonal projection onto the orthogonal complement of $\ker A_\nu(x,t)$, denoted $\ker A_\nu(x,t)^\perp$, defined by

$$P(x,t) = \frac{1}{2\pi i} \int_{C(x,t)} (\lambda - A_\nu(x,t))^{-1} d\lambda, \quad (x,t) \in \Sigma_\infty,$$

where $C(x,t)$ is a closed rectifiable Jordan curve with positive orientation in the complex plane, enclosing all and only all the non-zero eigenvalues of $A_\nu(x,t)$. $P(x,t)$ is the sum of eigenprojections corresponding to the non-zero eigenvalues of $A_\nu(x,t)$. Given an arbitrary smooth extension on $\overline{Q}_\infty$, denoted by the same symbol P, then Pu is the so-called *noncharacteristic* component of u, while $(I-P)u$ is the so-called *characteristic* component of u.

We study the problem (1)-(3) under the following assumptions.

(A) L is *Friedrichs symmetrizable*, namely there exists a matrix S_0, definite positive and symmetric on $\overline{Q}_\infty$, and such that the matrices $S_0 A_i$, for $i = 1, \dots, n$, are also symmetric.

(B) The boundary is *characteristic with constant rank*, namely the boundary matrix A_ν is singular on Σ_∞ and $0 < r := \operatorname{rank} A_\nu(x,t) < N$ for all $(x,t) \in \Sigma_\infty$; this assumption yields that the number of negative eigenvalues (counted with multiplicity) of A_ν is constant on Σ_∞.

(C) $M = M(x,t)$ is a $d \times N$ matrix-valued function of C^∞-class, and $d = \operatorname{rank} M(x,t)$ equals the number of negative eigenvalues of $A_\nu(x,t)$. Furthermore $\ker A_\nu(x,t) \subset \ker M(x,t)$, for all $(x,t) \in \Sigma_\infty$.

(D) The orthogonal projection $P(x,t)$ onto $\ker A_\nu(x,t)^\perp$, $(x,t) \in \Sigma_\infty$, is a matrix-valued function of C^∞-class on Σ_∞. We denote by the same symbol $P(x,t)$ an arbitrary smooth extension on $\overline{Q}_\infty$.

(E) *Existence of the L^2 weak solution for non-homogeneous boundary conditions.* Assume that $S_0, A_i \in Lip(\overline{Q}_\infty)$, for $i = 1, \dots, n$. For all $T > 0$ and all matrices $B \in L^\infty(\overline{Q}_T)$, there exist constants $\gamma_0 \geq 1$ and $C_0 > 0$ such that for all $F \in L^2(Q_T)$, $G \in L^2(\Sigma_T)$, $f \in L^2(\Omega)$ the problem (1)-(3), with data (F, G, f), admits a unique solution $u \in L^2(Q_T)$ such that $Pu_{|\,\Sigma_T} \in L^2(\Sigma_T)$. Furthermore

$u \in C([0,T];L^2(\Omega))$, and it satisfies an a priori estimate of the form

$$\begin{aligned} &e^{-2\gamma\tau}||u(\tau)||^2_{L^2(\Omega)} + \gamma\int_0^\tau e^{-2\gamma t}||u(t)||^2_{L^2(\Omega)}\,dt + \int_0^\tau e^{-2\gamma t}||Pu_{|\partial\Omega}(t)||^2_{L^2(\partial\Omega)}\,dt \\ &\le C_0\left(||f||^2_{L^2(\Omega)} + \frac{1}{\gamma}\int_0^\tau e^{-2\gamma t}||F(t)||^2_{L^2(\Omega)}\,dt + \int_0^\tau e^{-2\gamma t}||G(t)||^2_{L^2(\partial\Omega)}\,dt\right), \end{aligned} \tag{5}$$

for all $\gamma \ge \gamma_0$ and $0 < \tau \le T$.

When an IBVP admits an apriori estimate of type (5), with $F = Lu$, $G = Mu$, for all $\tau > 0$ and all sufficiently smooth functions u, one says that the IBVP is *strongly L^2 well posed*, see e.g. [4]. A necessary condition for (5) is the validity of the *uniform* Kreiss-Lopatinski condition (UKL). (An estimate of this form has been obtained by Rauch [21].) On the other hand, (UKL) is not sufficient for the well-posedness and other structural assumptions have to be taken, see [4].

When dealing with homogeneous boundary conditions, it happens that one can prove the existence of the L^2 solution without the direct L^2 control of the trace of Pu, as in (5). The boundary condition has to be intended in the $H^{-1/2}$ sense. For example, this happens with maximally nonnegative boundary conditions, which are not strictly dissipative. For this case we may assume a weaker version of (E), as follows. Observe that (F) is weaker than (E) for $G = 0$, because it provides no control on the trace of Pu.

(F) *Existence of the solution for homogeneous boundary conditions.* Assume that $S_0, A_i \in Lip(\overline{Q}_\infty)$, for $i = 1,\dots,n$. For all $T > 0$ and all matrices $B \in L^\infty(\overline{Q}_T)$, there exist constants $\gamma_0 \ge 1$ and $C_0 > 0$ such that for all $F \in L^2(Q_T)$, $f \in L^2(\Omega)$ the problem (1)-(3), with data $(F, G = 0, f)$, admits a unique solution $u \in C([0,T];L^2(\Omega))$, and it satisfies an a priori estimate of the form

$$e^{-2\gamma\tau}||u(\tau)||^2_{L^2(\Omega)} + \gamma\int_0^\tau e^{-2\gamma t}||u(t)||^2_{L^2(\Omega)}\,dt \le C_0\left(||f||^2_{L^2(\Omega)} + \frac{1}{\gamma}\int_0^\tau e^{-2\gamma t}||F(t)||^2_{L^2(\Omega)}\,dt\right), \tag{6}$$

for all $\gamma \ge \gamma_0$ and $0 < \tau \le T$.

Finally, we add the following technical assumption that for C^∞ approximations of (1)-(3) one still has the existence of L^2 solutions. This stability property holds true for maximally nonnegative boundary conditions and for uniform KL conditions.

(G) Given matrices $(S_0, A_i, B) \in \mathcal{C}_T(H^\sigma_*) \times \mathcal{C}_T(H^\sigma_*) \times \mathcal{C}_T(H^{\sigma-2}_*)$, where $\sigma \ge [(n+1)/2] + 4$, enjoying properties (A) - (E) (or (A) - (D), (F)), let $(S_0^{(k)}, A_i^{(k)}, B^{(k)})$ be C^∞ matrix-valued functions converging to (S_0, A_i, B) in $\mathcal{C}_T(H^\sigma_*) \times \mathcal{C}_T(H^\sigma_*) \times \mathcal{C}_T(H^{\sigma-2}_*)$ as $k \to \infty$, and satisfying properties (A) - (D). Then, for k sufficiently large, property (E) (resp. (F)) holds also for the approximating problems with coefficients $(S_0^{(k)}, A_i^{(k)}, B^{(k)})$.

Following Rauch [22, Theorem 7], one can prove that, for any $u \in L^2(Q_T)$ with $Lu \in L^2(Q_T)$ the trace of $A_\nu u$ on Σ_T exists in $H^{-1/2}(\Sigma_T)$. In the same way the restrictions of u to $\Omega \times \{t = 0\}$ and $\Omega \times \{t = T\}$ are well defined elements of $H^{1/2}(\Omega)'$.

The solution of (1)-(3), considered in statements (E), (F) is intended in the following sense: for all $v \in H^1(Q_T)$ such that $v_{|\,\Sigma_T} \in (A_\nu(\ker M))^\perp$ and $v(\cdot, T) = 0$ in Ω, there holds:

$$\int_{Q_T}\langle u, L^*v\rangle\,dx\,dt = \int_{Q_T}\langle F, v\rangle\,dx\,dt - \int_{\Sigma_T}\langle A_\nu g, v\rangle\,d\sigma_x\,dt + \int_\Omega\langle f, v(0)\rangle\,dx,$$

where L^* is the adjoint operator of L and g is a function defined on Σ_T such that $Mg = G$.

Remark 1. Boundary condition. *For a given boundary matrix $M(x,t)$ satisfying assumption* (C)*, there exists a matrix $M_0(x,t)$ such that $M(x,t) = M_0(x,t)A_\nu(x,t)$ for every $(x,t) \in \Sigma_T$. Therefore, for L^2solutions of* (1) *one has*

$$Mu = G \quad on\ \Sigma_T \iff M_0 A_\nu u_{|\Sigma_T} = G \quad on\ \Sigma_T. \tag{7}$$

In order to study the regularity of solutions to the IBVP (1)-(3), we need to impose some *compatibility conditions* on the data F, G, f. The compatibility conditions are defined in the usual way, see [24]. Given the IBVP (1)-(3), we recursively define $f^{(h)}$ by formally taking $h-1$ time derivatives of $Lu = F$, solving for $\partial_t^h u$ and evaluating it at $t = 0$. For $h = 0$ we set $f^{(0)} := f$. The *compatibility condition* of order $k \geq 0$ for the IBVP (1)-(3) reads as

$$\sum_{h=0}^{p} \left(\begin{array}{c} p \\ h \end{array} \right) (\partial_t^{p-h} M)_{|\, t=0} f^{(h)} = \partial_t^p G_{|\, t=0} \,, \quad \text{on } \partial\Omega \,,\ p = 0, \dots, k \,. \tag{8}$$

The aim of the paper is to prove the following theorems. The square brackets [ ] denote the integer part.

Theorem 2. *Let $m \in \mathbb{N}$ and $s = \max\{m, \left[(n+1)/2\right] + 5\}$. Assume that $S_0, A_j, \in \mathcal{C}_T(H^s_*)$, for $j = 1, \dots, n$, and $B \in \mathcal{C}_T(H^{s-1}_*)$ (or $B \in \mathcal{C}_T(H^s_*)$ if $m = s$). Assume also that problem* (1)-(3) *obeys the assumptions* (A)-(E), (G). *Then for all $F \in H^m_*(Q_T)$, $G \in H^m(\Sigma_T)$, $f \in H^m_*(\Omega)$, with $f^{(h)} \in H^{m-h}_*(\Omega)$ for $h = 1, \dots, m$, satisfying the compatibility condition* (8) *of order $m-1$, the unique solution u to* (1)–(3), *with data (F, G, f), belongs to $\mathcal{C}_T(H^m_*)$ and $Pu_{|\, \Sigma_T} \in H^m(\Sigma_T)$. Moreover u satisfies the a priori estimate*

$$||u||_{\mathcal{C}_T(H^m_*)} + ||Pu_{|\, \Sigma_T}||_{H^m(\Sigma_T)} \leq C_m \left(|||f|||_{m,*} + ||F||_{H^m_*(Q_T)} + ||G||_{H^m(\Sigma_T)} \right) , \tag{9}$$

with a constant $C_m > 0$ depending only on A_j, B.

The next theorem covers the case when assumption (F) substitutes (E). Here we loose the control at the boundary of Pu, see however Remark 4.

Theorem 3. *Let $m \in \mathbb{N}$ and $s = \max\{m, \left[(n+1)/2\right] + 5\}$. Assume that $S_0, A_j, \in \mathcal{C}_T(H^s_*)$, for $j = 1, \dots, n$, and $B \in \mathcal{C}_T(H^{s-1}_*)$ (or $B \in \mathcal{C}_T(H^s_*)$ if $m = s$). Assume also that problem* (1)-(3) *obeys the assumptions* (A)-(D), (F), (G)*; then for all $F \in H^m_*(Q_T)$, $f \in H^m_*(\Omega)$, with $f^{(h)} \in H^{m-h}_*(\Omega)$, for $h = 1, \dots, m$, satisfying the compatibility condition* (8) *of order $m-1$ (with $G = 0$), the unique solution u to* (1)–(3), *with data $(F, G = 0, f)$, belongs to $\mathcal{C}_T(H^m_*)$. Moreover u satisfies the a priori estimate*

$$||u||_{\mathcal{C}_T(H^m_*)} \leq C'_m \left(|||f|||_{m,*} + ||F||_{H^m_*(Q_T)} \right) , \tag{10}$$

with a constant $C'_m > 0$ depending only on A_j, B.

For the function spaces involved in the statements above, and the norms appearing in (9), (10), we refer to the next Section 2.

Remark 4. *In case of homogeneous boundary conditions, when assumption* (E) *is substituted by* (F), *we get no direct information about the trace of the solution at the boundary. If $u \in \mathcal{C}_T(H^m_*)$, $m > 1$, the imbedding theorem $H^m_*(\Omega) \hookrightarrow H^{m-1}(\partial\Omega)$ (see* [19]*) allows to obtain $Pu_{|\, \Sigma_T} \in H^{m-1}(\Sigma_T)$. However, one can see that actually any solution $u \in \mathcal{C}_T(H^m_*)$ has some extra regularity. In fact, one can prove that $Pu \in \mathcal{C}_T(H^m_{**})$, see* [29, Theorem 4.2].

Because of the imbedding $H^m_{**}(\Omega) \hookrightarrow H^{m-1/2}(\partial\Omega)$ (see [29, 38]*), and the property $Mu = MPu$ (following from assumption* (C)*), we may infer $Pu_{|\, \Sigma_T} \in \mathcal{C}_T(H^{m-1/2}(\partial\Omega))$. This shows that, under the weaker assumption* (F), *the trace of the noncharacteristic component of u has a loss of half derivative at the boundary, w.r.t. the stronger case of assumption* (E) *(Theorem 2).*

On the other hand, in both cases (either nonhomogeneous with (E) *or homogeneous with* (F)*) there is no control at the boundary of the characteristic component $(I-P)u$. Thus we can only have $(I-P)u_{|\, \Sigma_T} \in \mathcal{C}_T(H^{m-1}(\partial\Omega))$ by the first imbedding theorem above, with the loss of one derivative.*

The paper is organized as follows. In Section 2 we introduce the function spaces and some notations. In Section 3 we give some technical results useful for the proof of the tangential regularity, discussed in Section 4. Sections 5 and 6 contain the proof of the normal regularity for $m = 1$ and $m \geq 2$, respectively. Many technical results are given in the Appendices. In particular, Appendix B contains new results about spaces H^m_*, which improve older results in the literature.

2. Function Spaces

For every integer $m \geq 1$, we denote by $H^m(\Omega)$, $H^m(Q_T)$ the usual Sobolev spaces of order m on Ω and Q_T respectively.
In order to define the anisotropic Sobolev spaces to be used in the sequel, first we need to introduce the differential operators in *tangential direction*. Let us denote

$$\mathbb{R}^n_+ := \{x = (x_1, x'),\ x_1 > 0,\ x' := (x_2, \dots, x_n) \in \mathbb{R}^{n-1}\}.$$

Throughout the paper, for every $j = 1, 2, \dots, n$ the differential operator Z_j is defined by

$$Z_1 := x_1 \partial_1\,, \quad Z_j := \partial_j\,,\ \text{for } j = 2, \dots, n\,.$$

Then, for every multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, the *tangential* (or *conormal*) differential operator Z^α is defined by setting

$$Z^\alpha := Z_1^{\alpha_1} \dots Z_n^{\alpha_n}$$

(we also write, with the standard multi-index notation, $\partial^\alpha = \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n}$).
Given an integer $m \geq 1$, we introduce the Sobolev space $H^m_{tan}(\Omega)$ in the following way. Let us take a covering $\{U_j\}_{j=0}^l$ of Ω as follows: first we cover $\partial\Omega$ by coordinate patches U_j, $j = 1, \dots, l$, with coordinate systems

$$\chi_j : U_j \cap \overline{\Omega} \to \{x_1 \geq 0,\ |x| < 1\}\,,$$

such that $\chi_j(U_j \cap \partial\Omega) = \{x_1 = 0\,,\ |x| < 1\}$. Next we cover $\Omega \setminus \bigcup_{j=1}^l U_j$ by $U_0 \subset\subset \Omega$. Choose a partition of unity $\{\psi_j\}_{j=0}^l$ subordinate to the covering $\{U_j\}_{j=0}^l$. Then one says that a distribution u belongs to $H^m_{tan}(\Omega)$ if and only if $\psi_0 u \in H^m(\mathbb{R}^n)$ and $\psi_j u \in H^m_{tan}(\mathbb{R}^n_+)$, in local coordinates in U_j, for all $j = 1, \dots, l$, where

$$H^m_{tan}(\mathbb{R}^n_+) := \{w \in L^2(\mathbb{R}^n_+)\ :\ Z^\alpha w \in L^2(\mathbb{R}^n_+)\,,\ |\alpha| \leq m\}\,.$$

The tangential Sobolev space $H^m_{tan}(\Omega)$ is equipped with the norm

$$||u||^2_{H^m_{tan}(\Omega)} := ||\psi_0 u||^2_{H^m(\mathbb{R}^n)} + \sum_{j=1}^l ||\psi_j u||^2_{H^m_{tan}(\mathbb{R}^n_+)} \tag{11}$$

where

$$||w||^2_{H^m_{tan}(\mathbb{R}^n_+)} := \sum_{|\alpha| \leq m} ||Z^\alpha w||^2_{L^2(\mathbb{R}^n_+)}\,.$$

Note that the definition of $H^m_{tan}(\Omega)$ does not depend on the choice of $U_j\,, \chi_j, \psi_j$, and that the norms arising from different choices of U_j, χ_j, ψ_j are equivalent. The same space is sometimes called *conormal Sobolev space* w.r.t. $\partial\mathbb{R}^n_+$ and denoted $H^m(\mathbb{R}^n_+; \partial\mathbb{R}^n_+)$, see e.g. [17].
Keeping the same notations used just above, for every positive integer m the Sobolev anisotropic space $H^m_*(\Omega)$ is defined to be the set of distributions u in Ω such that $\psi_0 u \in H^m(\mathbb{R}^n)$ and $\psi_j u \in H^m_*(\mathbb{R}^n_+)$, in local coordinates in U_j, for all $j = 1, \dots, l$, where

$$H^m_*(\mathbb{R}^n_+) := \{w \in L^2(\mathbb{R}^n_+)\ : \quad Z^\alpha \partial_1^k w \in L^2(\mathbb{R}^n_+)\,,\ |\alpha| + 2k \leq m\}\,.$$

$H^m_*(\Omega)$ is equipped with the norm

$$||u||^2_{H^m_*(\Omega)} := ||\psi_0 u||^2_{H^m(\mathbb{R}^n)} + \sum_{j=1}^l ||\psi_j u||^2_{H^m_*(\mathbb{R}^n_+)}\,, \tag{12}$$

where

$$||w||^2_{H^m_*(\mathbb{R}^n_+)} := \sum_{|\alpha|+2k \leq m} ||Z^\alpha \partial_1^k w||^2_{L^2(\mathbb{R}^n_+)}\,.$$

We also define the Sobolev anisotropic space $H^m_{**}(\Omega)$ as the set of distributions u in Ω such that $\psi_0 u \in H^m(\mathbb{R}^n)$ and $\psi_j u \in H^m_{**}(\mathbb{R}^n_+)$, in local coordinates in U_j, for all $j = 1, \dots, l$, where

$$H^m_{**}(\mathbb{R}^n_+) := \{w \in L^2(\mathbb{R}^n_+)\ : \quad Z^\alpha \partial_1^k w \in L^2(\mathbb{R}^n_+)\,,\ |\alpha| + 2k \leq m+1,\ |\alpha| \leq m\}\,.$$

$H^m_{**}(\Omega)$ is equipped with the norm

$$||u||^2_{H^m_{**}(\Omega)} := ||\psi_0 u||^2_{H^m(\mathbb{R}^n)} + \sum_{j=1}^{l} ||\psi_j u||^2_{H^m_{**}(\mathbb{R}^n_+)}\,, \tag{13}$$

where

$$||w||^2_{H^m_{**}(\mathbb{R}^n_+)} := \sum_{|\alpha|+2k\le m+1,\, |\alpha|\le m} ||Z^\alpha \partial_1^k w||^2_{L^2(\mathbb{R}^n_+)}\,.$$

The spaces $H^m_{tan}(\Omega)$, $H^m_*(\Omega)$, $H^m_{**}(\Omega)$, endowed with their norms (11), (12), (13) respectively, are Hilbert spaces. $C^\infty(\overline{\Omega})$ is dense in each of them. For an extensive study of the anisotropic spaces $H^m_*(\Omega)$ and $H^m_{**}(\Omega)$, we refer the reader to [19, 20, 29, 34, 38]. We observe that

$$\begin{array}{l} H^m(\Omega) \hookrightarrow H^m_{**}(\Omega) \hookrightarrow H^m_*(\Omega) \hookrightarrow H^m_{tan}(\Omega) \subset H^m_{loc}(\Omega)\,, \quad H^m_{**}(\Omega) \hookrightarrow H^{[(m+1)/2]}(\Omega)\,, \\ H^m_*(\Omega) \hookrightarrow H^{[m/2]}(\Omega)\,, \quad H^1_{**}(\Omega) = H^1(\Omega)\,, \quad H^1_*(\Omega) = H^1_{tan}(\Omega) \end{array} \tag{14}$$

(except for $H^m_{loc}(\Omega)$ all imbeddings are continuous). For the sake of convenience we also set $H^0_{**}(\Omega) = H^0_*(\Omega) = H^0_{tan}(\Omega) = L^2(\Omega)$. In a similar way we define the anisotropic spaces $H^m_{tan}(Q_T)$, $H^m_*(Q_T)$, equipped with their natural norms.

Let $C^j([0,T];X)$ denote the space of all X-valued j-times continuously differentiable functions of t, for $t \in [0,T]$. We denote by $W^{j,\infty}(0,T;X)$ the space of essentially bounded functions, together with the derivatives up to order j on $[0,T]$, with values in X. We define the spaces

$$\mathcal{C}_T(H^m_*) := \bigcap_{j=0}^{m} C^j([0,T]; H^{m-j}_*(\Omega))\,, \quad \mathcal{L}^\infty_T(H^m_*) := \bigcap_{j=0}^{m} W^{j,\infty}(0,T; H^{m-j}_*(\Omega))\,,$$

with norm

$$||u||^2_{\mathcal{C}_T(H^m_*)} = ||u||^2_{\mathcal{L}^\infty_T(H^m_*)} := \sum_{j=0}^{m} \sup_{t\in[0,T]} ||\partial_t^j u(t)||^2_{H^{m-j}_*(\Omega)}\,.$$

The space $\mathcal{C}_T(H^m_{**})$ is defined in a similar way. For the initial data we set

$$|||f|||^2_{m,*} := \sum_{j=0}^{m} ||f^{(j)}||^2_{H^{m-j}_*(\Omega)}\,.$$

3. Preliminaries

In this section, we prove the tangential regularity in space-time of the weak solution. Throughout this section, the time t is allowed to span the whole real line; therefore, setting $x_{n+1} = t$, we have now $x = (x_1, \dots, x_n, x_{n+1}) \in \mathbb{R}^{n+1}_+ := \mathbb{R}^n_+ \times \mathbb{R}$. Following Nishitani and Takayama [17] we introduce an operator $\sharp$ sending $u \in L^2(\mathbb{R}^{n+1}_+)$ to $u^\sharp \in L^2(\mathbb{R}^{n+1})$ and a "tangential" mollifier J_ε so that $(J_\varepsilon u)^\sharp = \chi_\varepsilon * u^\sharp$, where $\chi_\varepsilon *$ is the classical mollifier in $\mathbb{R}^{n+1}$. Using J_ε we follow the same lines in Tartakoff [40], Nishitani and Takayama [17] to get regularity of weak solution u.

Let us introduce the maps $\sharp : L^2(\mathbb{R}^{n+1}_+) \to L^2(\mathbb{R}^{n+1})$ and $\natural : L^\infty(\mathbb{R}^{n+1}_+) \to L^\infty(\mathbb{R}^{n+1})$ by

$$w^\sharp(x) := w(e^{x_1}, x')e^{x_1/2}, \quad a^\natural(x) = a(e^{x_1}, x'). \tag{15}$$

Both are norm preserving bijections and it is easy to see that

$$(aw)^\sharp = a^\natural w^\sharp\,, \tag{16}$$

$$\partial_j(a^\natural) = (Z_j a)^\natural, \quad j = 1, \dots, n+1\,, \tag{17}$$

$$\partial_1(w^\sharp) = (Z_1 w)^\sharp + \frac{1}{2} w^\sharp\,, \tag{18}$$

$$\partial_j(w^\sharp) = (Z_j w)^\sharp\,, \quad j = 2, \dots, n+1. \tag{19}$$

It can be also proved that the map

$$\sharp : H^k_{tan}(\mathbb{R}^{n+1}_+) \to H^k(\mathbb{R}^{n+1}) \tag{20}$$

is a topological isomorphism.
We consider the following family of norms

$$||w||^2_{\mathbb{R}^{n+1}_+,k,tan,\delta} := ||w^\sharp||^2_{\mathbb{R}^{n+1},k,\delta} := \int_{\mathbb{R}^{n+1}} |(w^\sharp)^\wedge(\xi)|^2 \langle\xi\rangle^{2(k+1)}\langle\delta\xi\rangle^{-2}d\xi, \tag{21}$$

for $0<\delta\le 1$, with $\langle\xi\rangle^2 := 1+|\xi|^2$. Here $(w^\sharp)^\wedge(\xi)$ denotes the Fourier transform of $w^\sharp(x)$ with respect to x. Note that this norm is equivalent to $||w||_{H^k_{tan}(\mathbb{R}^{n+1}_+)}$ for each fixed δ, $0<\delta\le 1$. When $\delta=1$ we write

$$||u||_{\mathbb{R}^{n+1}_+,k,tan} := ||u||_{\mathbb{R}^{n+1}_+,k,tan,1}.$$

Moreover, the following characterization of the tangential Sobolev spaces $H^k_{tan}(\mathbb{R}^{n+1}_+)$ can be proved (cf.[17]).

Lemma 5. *$u\in H^k_{tan}(\mathbb{R}^{n+1}_+)$ if and only if $u\in H^{k-1}_{tan}(\mathbb{R}^{n+1}_+)$ and the norm $||u||_{\mathbb{R}^{n+1}_+,k-1,tan,\delta}$ remains bounded when $\delta\downarrow 0$. In this case, we have*

$$||u||_{\mathbb{R}^{n+1}_+,k-1,tan,\delta}\uparrow ||u||_{\mathbb{R}^{n+1}_+,k,tan}\,, \qquad \textit{as } \delta\downarrow 0.$$

3.1. **Tangential mollifiers.** Following Nishitani and Takayama [17] we introduce suitable mollifiers, well suited to the tangential Sobolev spaces.

Let χ be a function in $C^\infty_0(\mathbb{R}^{n+1})$. For all $0<\varepsilon<1$ set $\chi_\varepsilon(y):=\varepsilon^{-(n+1)}\chi(y/\varepsilon)$. We define $J_\varepsilon : L^2(\mathbb{R}^{n+1}_+)\to L^2(\mathbb{R}^{n+1}_+)$ by

$$J_\varepsilon w(x) := \int_{\mathbb{R}^{n+1}} w(x_1e^{-y_1}, x'-y')e^{-y_1/2}\chi_\varepsilon(y)dy\,, \tag{22}$$

which differs from the one introduced in Rauch [22] by the factor $e^{-y_1/2}$. Using (15), the following identity can be easily proven

$$(J_\varepsilon w)^\sharp(x) = \chi_\varepsilon * w^\sharp(x) \tag{23}$$

for all $w\in L^2(\mathbb{R}^{n+1}_+)$ and $0<\varepsilon<1$. A combination of (20), (23) and the known properties of the convolution by χ_ε then gives

$$\exists\, C>0:\ \ ||J_\varepsilon w||_{L^2(\mathbb{R}^{n+1}_+)}\le C||w||_{L^2(\mathbb{R}^{n+1}_+)}, \quad \forall\varepsilon>0; \tag{24}$$

$$\begin{aligned}&\forall k\ge 1\ \ \exists\, C>0:\ \ \forall w\in L^2(\mathbb{R}^{n+1}_+), J_\varepsilon w\in H^k_{tan}(\mathbb{R}^{n+1}_+)\,\text{with}\\ &\qquad ||J_\varepsilon w||_{H^k_{tan}(\mathbb{R}^{n+1}_+)}\le \frac{C}{\varepsilon^k}||w||_{L^2(\mathbb{R}^{n+1}_+)}, \quad \forall\varepsilon>0.\end{aligned} \tag{25}$$

Moreover, we have that

$$[Z_j, J_\varepsilon] = 0\,, \quad j=1,\dots,n+1. \tag{26}$$

Pulling back w by $\sharp$ and applying Theorem 2.4.1 in Hörmander [13] to $w^\sharp$, we get

Proposition 6. *Assume that the function $\chi\in C^\infty_0(\mathbb{R}^{n+1})$ satisfies*

$$\widehat{\chi}(\xi) = O(|\xi|^p) \quad \textit{as } \xi\to 0, \quad \textit{for some } p\in\mathbb{N}; \tag{27}$$

$$\widehat{\chi}(t\xi)=0\,, \quad \textit{for all } t\in\mathbb{R}\,, \quad \textit{implies } \xi=0. \tag{28}$$

Then for $k\in\mathbb{N}$ with $k<p$, there exists $C_0=C_0(\chi,k)>0$ such that

$$\begin{aligned}&C_0^{-1}||w||^2_{\mathbb{R}^{n+1}_+,k-1,tan,\delta}\\ &\le \int_0^1 ||J_\varepsilon w||^2_{L^2(\mathbb{R}^{n+1}_+)}\varepsilon^{-2k}\left(1+\tfrac{\delta^2}{\varepsilon^2}\right)^{-1}\tfrac{d\varepsilon}{\varepsilon} + ||w||^2_{\mathbb{R}^{n+1}_+,k-1,tan}\\ &\le C_0||w||^2_{\mathbb{R}^{n+1}_+,k-1,tan,\delta},\end{aligned} \tag{29}$$

for all $0<\delta\le 1$ and $w\in H^{k-1}_{tan}(\mathbb{R}^{n+1}_+)$. Moreover, the second inequality in (29) remains true also if only (27) is satisfied.

Combining the characterization of tangential Sobolev spaces given by Lemma 5 and Proposition 6, we get the following characterization of Sobolev spaces $H^k_{tan}(\mathbb{R}^{n+1}_+)$ by means of J_ε, which will be used later.

Proposition 7. *Assume that $\chi \in C_0^\infty(\mathbb{R}_+^{n+1})$ satisfies assumptions (27), (28). Then for all $k \in \mathbb{N}$ with $k < p$, we have that $u \in H_{tan}^k(\mathbb{R}_+^{n+1})$ if and only if*

a. $u \in H_{tan}^{k-1}(\mathbb{R}_+^{n+1})$;

b. $\int_0^1 ||J_\varepsilon u||^2_{L^2(\mathbb{R}_+^{n+1})} \varepsilon^{-2k} \left(1 + \frac{\delta^2}{\varepsilon^2}\right)^{-1} \frac{d\varepsilon}{\varepsilon}$ *is uniformly bounded for $0 < \delta \leq 1$.*

3.2. **Estimate of commutators.** The next lemma shows that part of the commutators $[L, J_\varepsilon]$, that will appear in the forthcoming analysis of Section 4, can be written as a sum of integral operators. We will denote by $C_{(0)}^\infty(\mathbb{R}_+^{n+1})$ the resctriction onto $\mathbb{R}_+^{n+1}$ of functions in $C_0^\infty(\mathbb{R}^{n+1})$.

Lemma 8. *Assume that $\chi \in C_0^\infty(\mathbb{R}^{n+1})$. If $u \in L^2(\mathbb{R}_+^{n+1})$ and if $a(x) \in C_{(0)}^\infty(\mathbb{R}_+^{n+1})$, then $([a, J_\varepsilon]u)^\sharp$ can be written in the form*

$$\int_{\mathbb{R}^{n+1}} b(x,y) \cdot y \, u^\sharp(x-y)\chi_\varepsilon(y)dy. \tag{30}$$

For $j = 1, \cdots, n+1$, $([aZ_j, J_\varepsilon]u)^\sharp$ can be written as a sum of terms of the form

$$\int_{\mathbb{R}^{n+1}} b(x,y) u^\sharp(x-y)\chi_\varepsilon(y)dy, \tag{31}$$

and

$$\frac{1}{\varepsilon}\int_{\mathbb{R}^{n+1}} b(x,y) \cdot y \, u^\sharp(x-y)(\partial_j\chi)_\varepsilon(y)dy. \tag{32}$$

Here $b(x,y) \in \mathcal{B}^\infty(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1})$, the set of all smooth functions on $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ with bounded derivatives of all orders.

Proof. Step 1: Let us consider $([a, J_\varepsilon]u)^\sharp$. By definition we get

$$([a, J_\varepsilon]u)^\sharp = a^\natural(u^\sharp * \chi_\varepsilon) - (a^\natural u^\sharp) * \chi_\varepsilon = \int_{\mathbb{R}^{n+1}} [a^\natural(x) - a^\natural(x-y)] u^\sharp(x-y)\chi_\varepsilon(y)dy.$$

We notice that we can write

$$a^\natural(x) - a^\natural(x-y) = \int_0^1 \sum_{i=1}^{n+1} \frac{\partial a^\natural}{\partial x_i}(x-(1-t)y)y_i dt = \sum_{i=1}^{n+1} b_i(x,y)y_i, \tag{33}$$

with $b_i(x,y) \in \mathcal{B}^\infty(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1})$, which gives (30).
Step 2: Let us consider $([aZ_j, J_\varepsilon]u)^\sharp$, $j = 1, \cdots, n+1$. By definition and (26) we can write

$$([aZ_j, J_\varepsilon]u)^\sharp = (aZ_jJ_\varepsilon u - J_\varepsilon(aZ_ju))^\sharp = (aJ_\varepsilon Z_j u - J_\varepsilon aZ_j u)^\sharp = ([a, J_\varepsilon]Z_j u)^\sharp .$$

For $j \geq 2$, noting that $(Z_ju)^\sharp(x-y) = -\frac{\partial}{\partial y_j} u^\sharp(x-y)$, we can write

$$\begin{aligned} ([a, J_\varepsilon]Z_j u)^\sharp &= \int_{\mathbb{R}^{n+1}} [a^\natural(x) - a^\natural(x-y)](Z_ju)^\sharp(x-y)\chi_\varepsilon(y)dy \\ &= -\int_{\mathbb{R}^{n+1}} [a^\natural(x) - a^\natural(x-y)]\frac{\partial}{\partial y_j}(u^\sharp(x-y))\chi_\varepsilon(y)dy \\ &= -\int_{\mathbb{R}^{n+1}} \frac{\partial}{\partial y_j}(a^\natural(x-y))u^\sharp(x-y)\chi_\varepsilon(y)dy \\ &+\frac{1}{\varepsilon}\int_{\mathbb{R}^{n+1}} [a^\natural(x) - a^\natural(x-y)]u^\sharp(x-y)\left(\frac{\partial\chi}{\partial y_j}\right)_\varepsilon(y)dy. \end{aligned} \tag{34}$$

The first integral in the above equality is just of the form (31). For the second one, writing $a^\natural(x) - a^\natural(x-y)$ by (33) immediately gives a sum of terms of type (32). For $j = 1$ we recall that $(Z_1u)^\sharp = \partial_1 u^\sharp - \frac{1}{2}u^\sharp$. Hence, integrating by parts, we get

$$\begin{aligned} ([aZ_1, J_\varepsilon]u)^\sharp &= -\int_{\mathbb{R}^{n+1}} [a^\natural(x) - a^\natural(x-y)]\left[\frac{\partial}{\partial y_1}u^\sharp(x-y) + \frac{1}{2}u^\sharp(x-y)\right]\chi_\varepsilon(y)dy \\ &= -\int_{\mathbb{R}^{n+1}} \frac{\partial}{\partial y_1}(a^\natural(x-y))u^\sharp(x-y)\chi_\varepsilon(y)dy - \frac{1}{2}\int_{\mathbb{R}^{n+1}}[a^\natural(x) - a^\natural(x-y)]u^\sharp(x-y)\chi_\varepsilon(y)dy \\ &+\frac{1}{\varepsilon}\int_{\mathbb{R}^{n+1}} [a^\natural(x) - a^\natural(x-y)]u^\sharp(x-y)\left(\frac{\partial\chi}{\partial y_1}\right)_\varepsilon(y)dy. \end{aligned}$$

The first two integrals in the above equality are just of the form (31). For the third one the same argument used in *Step 1* shows that it has the form (32). ☐

The following Lemma is due to Nishitani and Takayama [17, Lemma 9.2].

Lemma 9. *Assume that $\chi \in C_0^\infty(\mathbb{R}^{n+1})$ satisfies (27), and let $a(x,y) \in \mathcal{B}^\infty(\mathbb{R}^{n+1}\times\mathbb{R}^{n+1})$. For $\alpha \in \mathbb{N}^{n+1}$ and $k \geq 1$ there is a constant $C = C(\chi, k, a, \alpha) > 0$ with the following property: if $u \in H^{k-1}_{tan}(\mathbb{R}^{n+1}_+)$ and if we set*

$$U_\varepsilon(x) = \int_{\mathbb{R}^{n+1}} a(x,y)\, u^\sharp(x-y)\, y^\alpha \chi_\varepsilon(y)dy$$

then for all $0 < \delta \leq 1$ we have

$$\int_0^1 ||U_\varepsilon||^2_{L^2(\mathbb{R}^{n+1})}\varepsilon^{-2k}\left(1+\frac{\delta^2}{\varepsilon^2}\right)^{-1}\frac{d\varepsilon}{\varepsilon} \leq \begin{cases} C||u||^2_{\mathbb{R}^{n+1}_+,k-1,tan,\delta} & \text{if } |\alpha| = 0, \\ C||u||^2_{H^{k-|\alpha|}_{tan}(\mathbb{R}^{n+1}_+)} & \text{if } 1 \leq |\alpha| \leq k, \\ C||u||^2_{L^2(\mathbb{R}^{n+1}_+)} & \text{if } |\alpha| \geq k+1. \end{cases} \tag{35}$$

Following the proof of Lemma 9 given in [17], we notice that the second inequality in (35) follows from the sharper one

$$\int_0^1 ||U_\varepsilon||^2_{L^2(\mathbb{R}^{n+1})}\varepsilon^{-2k}\left(1+\frac{\delta^2}{\varepsilon^2}\right)^{-1}\frac{d\varepsilon}{\varepsilon} \leq C||u||^2_{\mathbb{R}^{n+1}_+,k-|\alpha|-1,tan,\delta} \quad \text{if } 1 \leq |\alpha| \leq k. \tag{36}$$

Hereafter we will also use this sharper inequality (36).

By Lemma 8 and Lemma 9 we can easily derive the following result.

Lemma 10. *Assume that $\chi \in C_0^\infty(\mathbb{R}^{n+1})$ satisfies (27) and let $a \in C^\infty_{(0)}(\mathbb{R}^{n+1}_+)$. Assume that $u \in H^{k-1}_{tan}(\mathbb{R}^{n+1}_+)$, $k \geq 1$. Then there exists a constant $C > 0$ such that for all $0 < \delta \leq 1$ the following hold*

$$\int_0^1 ||[a, J_\varepsilon]u||^2_{L^2(\mathbb{R}^{n+1}_+)}\varepsilon^{-2k}\left(1+\frac{\delta^2}{\varepsilon^2}\right)^{-1}\frac{d\varepsilon}{\varepsilon} \leq C||u||^2_{\mathbb{R}^{n+1}_+,k-2,tan,\delta}\,,$$

and, for $j = 1, \cdots, n+1$,

$$\int_0^1 ||[aZ_j, J_\varepsilon]u||^2_{L^2(\mathbb{R}^{n+1}_+)}\varepsilon^{-2k}\left(1+\frac{\delta^2}{\varepsilon^2}\right)^{-1}\frac{d\varepsilon}{\varepsilon} \leq C||u||^2_{\mathbb{R}^{n+1}_+,k-1,tan,\delta}.$$

Proof. By Lemma 8 we know that

$$||[a, J_\varepsilon]u||_{L^2(\mathbb{R}^{n+1}_+)} = ||([a, J_\varepsilon]u)^\sharp||_{L^2(\mathbb{R}^{n+1})} = ||U_\varepsilon||_{L^2(\mathbb{R}^{n+1})}$$

where U_ε is a function as in Lemma 9, with $|\alpha| = 1$. Hence we get the first estimate by applying (36). Now we consider $([aZ_j, J_\varepsilon]u)^\sharp$. By Lemma 8, for every $j = 1, \cdots, n+1$, this term can be written as a sum of terms

$$\int_{\mathbb{R}^{n+1}} b_0(x,y)u^\sharp(x-y)\chi_\varepsilon(y)dy + \sum_{i=1}^{n+1}\frac{1}{\varepsilon}\int_{\mathbb{R}^{n+1}} b_i(x,y)u^\sharp(x-y)y_i\left(\partial_j\chi\right)_\varepsilon(y)dy.$$

The first integral is estimated using Lemma 9 with $|\alpha| = 0$. For the second one we denote

$$V(x) = \sum_{i=1}^{n+1}\frac{1}{\varepsilon}\int_{\mathbb{R}^{n+1}} b_i(x,y)u^\sharp(x-y)y_i\left(\partial_j\chi\right)_\varepsilon(y)dy.$$

Then, for all $0 < \delta \leq 1$ we get

$$\begin{aligned} &\int_0^1 ||V||^2_{L^2(\mathbb{R}^{n+1})}\varepsilon^{-2k}\left(1+\frac{\delta^2}{\varepsilon^2}\right)^{-1}\frac{d\varepsilon}{\varepsilon} \\ &\leq \sum_{i=1}^{n+1}\int_0^1 \left\|\int_{\mathbb{R}^{n+1}} b_i(.,y)u^\sharp(.-y)y_i\left(\partial_j\chi\right)_\varepsilon dy\right\|^2_{L^2(\mathbb{R}^{n+1})}\varepsilon^{-2(k+1)}\left(1+\frac{\delta^2}{\varepsilon^2}\right)^{-1}\frac{d\varepsilon}{\varepsilon}. \end{aligned} \tag{37}$$

Each term in the sum appearing in the right-hand side of the last inequality is of the form

$$\int_0^1 ||U_\varepsilon||^2_{L^2(\mathbb{R}^{n+1})}\varepsilon^{-2(k+1)}\left(1+\frac{\delta^2}{\varepsilon^2}\right)^{-1}\frac{d\varepsilon}{\varepsilon},$$

with a suitable function U_ε as the one considered in Lemma 9, with $|\alpha|=1$ and $\partial_j\chi$ instead of χ. We notice that $\partial_j\chi$ still satisfies (27), with $p+1$ instead of p. Hence, applying (36) with $k+1$ instead of k, we obtain that the right-hand side of (37) can be bounded by $C||u||^2_{\mathbb{R}^{n+1}_+,k-1,tan,\delta}$. Collecting all the previous estimates we derive the second inequality of Lemma 10. □

As a consequence of Proposition 6, from Lemma 10 we derive the following.

Corollary 11. *Assume that $\chi\in C_0^\infty(\mathbb{R}^{n+1})$ satisfies (27), (28), and let $a\in C^\infty_{(0)}(\mathbb{R}^{n+1}_+)$. Then for all $k\geq 1$ and $u\in H^{k-1}_{tan}(\mathbb{R}^{n+1}_+)$, there exists a constant $C>0$, such that for all $0<\delta\leq 1$*

$$\begin{aligned}&\textstyle\int_0^1 ||[a,J_\varepsilon]u||^2_{L^2(\mathbb{R}^{n+1}_+)}\varepsilon^{-2k}\left(1+\frac{\delta^2}{\varepsilon^2}\right)^{-1}\frac{d\varepsilon}{\varepsilon}\\ &\leq C\left(\textstyle\int_0^1 ||J_\varepsilon u||^2_{L^2(\mathbb{R}^{n+1}_+)}\varepsilon^{-2k+2}\left(1+\frac{\delta^2}{\varepsilon^2}\right)^{-1}\frac{d\varepsilon}{\varepsilon}+||u||^2_{\mathbb{R}^{n+1}_+,k-2,tan}\right),\end{aligned}$$

and, for $j=1,\dots,n+1$,

$$\begin{aligned}&\textstyle\int_0^1 ||[aZ_j,J_\varepsilon]u||^2_{L^2(\mathbb{R}^{n+1}_+)}\varepsilon^{-2k}\left(1+\frac{\delta^2}{\varepsilon^2}\right)^{-1}\frac{d\varepsilon}{\varepsilon}\\ &\leq C\left(\textstyle\int_0^1 ||J_\varepsilon u||^2_{L^2(\mathbb{R}^{n+1}_+)}\varepsilon^{-2k}\left(1+\frac{\delta^2}{\varepsilon^2}\right)^{-1}\frac{d\varepsilon}{\varepsilon}+||u||^2_{\mathbb{R}^{n+1}_+,k-1,tan}\right).\end{aligned}$$

3.3. **Estimate of boundary terms.** For the sequel, it is convenient to assume for χ the following explicit form

$$\chi(y)=(-\Delta)^{\frac{p}{2}}\phi(y), \tag{38}$$

with a given function $\phi\in C_0^\infty(\mathbb{R}^{n+1})$ satifying $\widehat{\phi}(0)=\int_{\mathbb{R}^{n+1}}\phi(y)dy\neq 0$; for simplicity, assume also that p is even. It follows that $\chi\in C_0^\infty(\mathbb{R}^{n+1})$ verifies (27), (28). Assume also that $\operatorname{supp}\phi\subset\{x_{n+1}>0\}$, so that also

$$\operatorname{supp}\chi\subset\{x_{n+1}>0\}. \tag{39}$$

Given this function χ, for $0<\varepsilon<1$, we define $\widetilde{\chi}_\varepsilon$ by formula

$$\widetilde{\chi}_\varepsilon(y'):=\int_{\mathbb{R}}e^{-y_1/2}\chi_\varepsilon(y_1,y')dy_1=\frac{1}{\varepsilon^n}\int_{\mathbb{R}}e^{-\varepsilon y_1/2}\chi\left(y_1,\frac{y'}{\varepsilon}\right)dy_1\,,\quad y'\in\mathbb{R}^n\,. \tag{40}$$

Let us emphasize that the functions $\widetilde{\chi}_\varepsilon$ cannot be written as

$$\widetilde{\chi}_\varepsilon(y')=\frac{1}{\varepsilon^n}\widetilde{\chi}\left(\frac{y'}{\varepsilon}\right),$$

for some $\widetilde{\chi}\in C_0^\infty(\mathbb{R}^n)$, because of the factor $e^{-\frac{\varepsilon y_1}{2}}$ appearing in the integral (40). Hence Theorem 2.4.1 in [13] cannot be directly applied. However, the proof of this theorem can be adapted to obtain the following

Proposition 12. *Let χ be a function in $C_0^\infty(\mathbb{R}^{n+1})$ satisfying all the preceding assumptions. Then for every $0<k<p$, there exists a positive constant $C=C(\chi,k,p)>0$ such that for all $u\in H^{k-1}(\mathbb{R}^n)$ the estimates*

$$\begin{aligned}&C^{-1}||u||^2_{\mathbb{R}^n,k-1,\delta}\\ &\leq\int_0^1||u*\widetilde{\chi}_\varepsilon||^2_{L^2(\mathbb{R}^n)}\varepsilon^{-2k}\left(1+\frac{\delta^2}{\varepsilon^2}\right)^{-1}\frac{d\varepsilon}{\varepsilon}+||u||^2_{\mathbb{R}^n,k-1}\\ &\leq C||u||^2_{\mathbb{R}^n,k-1,\delta},\end{aligned} \tag{41}$$

hold for all $0<\delta\leq 1$.

The proof of Proposition 12 is postponed to Appendix A.
We recall the following result.

Lemma 13. *$u \in H^k(\mathbb{R}^n)$ if and only if $u \in H^{k-1}(\mathbb{R}^n)$ and the norm $||u||_{\mathbb{R}^n,k-1,\delta}$ remains bounded when $\delta \downarrow 0$. In this case, we have*

$$||u||_{\mathbb{R}^n,k-1,\delta} \uparrow ||u||_{\mathbb{R}^n,k}\,, \quad \text{as } \delta \downarrow 0.$$

The following characterization of Sobolev spaces in $\mathbb{R}^n$ follows from Proposition 12 and Lemma 13.

Proposition 14. *Assume that $\chi \in C_0^\infty(\mathbb{R}^{n+1})$ obeys the assumptions of Proposition 12. Then for all real $0 < k < p$, we have that $u \in H^k(\mathbb{R}^n)$ if and only if*

a. $u \in H^{k-1}(\mathbb{R}^n)$;

b. $\int_0^1 ||w * \widetilde{\chi}_\varepsilon||^2_{L^2(\mathbb{R}^n)}\varepsilon^{-2k}\left(1+\frac{\delta^2}{\varepsilon^2}\right)^{-1}\frac{d\varepsilon}{\varepsilon}$ *are bounded uniformly for* $0 < \delta \leq 1$.

4. The homogeneous IBVP. Tangential regularity

We introduce the new unknown $u_\gamma(x,t) := e^{-\gamma t}u(x,t)$ and the new data $F_\gamma := e^{-\gamma t}F(x,t)$, $G_\gamma := e^{-\gamma t}G(x,t)$. Then problem (1)-(3) becomes equivalent to

$$\begin{array}{ll} L_\gamma u_\gamma = F_\gamma\,, & \text{in } Q_T,\\ Mu_\gamma = G_\gamma\,, & \text{on } \Sigma_T\\ u_{\gamma\,|t=0} = f\,, & \text{in } \Omega, \end{array} \tag{42}$$

with

$$L_\gamma := \gamma + L.$$

In this section we concentrate on the study of the tangential regularity of the solution to the IBVP (42), where the initial datum f is identically zero and the data F_γ and G_γ satisfy the compatibility conditions in a more restrictive form than (8). More precisely, we concentrate on the *homogeneous* IBVP

$$\begin{array}{ll} L_\gamma u_\gamma = F_\gamma\,, & \text{in } Q_T\,,\\ Mu_\gamma = G_\gamma\,, & \text{on } \Sigma_T\,,\\ u_{\gamma|\,t=0} = 0\,, & \text{in } \Omega\,. \end{array} \tag{43}$$

We remark that here and in the following the word *homogeneous* is referred by convention to the initial datum f and not to the boundary datum G, contrary to the terminology used in Section 1.

For a given integer $m \geq 1$, we assume that F_γ and G_γ satisfy the following conditions

$$\partial_t^h F_{\gamma|\,t=0} = 0\,, \quad \partial_t^h G_{\gamma|\,t=0} = 0\,, \quad h = 0,\ldots,m-1\,. \tag{44}$$

It is worth to notice that conditions (44) imply the compatibility conditions (8), in the case $f = 0$. We prove the following theorem for smooth coefficients. The general case with coefficients of finite regularity will be treated later on by a density argument.

Theorem 15. *Assume that A_i, B, for $i = 1,\ldots,n$, are in $C^\infty(\overline{Q}_\infty)$, and that problem (43) satisfies assumptions (A)–(E); then for all $T > 0$ and $m \in \mathbb{N}$ there exist constants $C_m > 0$ and γ_m, with $\gamma_m \geq \gamma_{m-1}$, such that for all $\gamma \geq \gamma_m$, for all $F_\gamma \in H^m_{tan}(Q_T)$ and all $G_\gamma \in H^m(\Sigma_T)$ satisfying (44) the unique solution u_γ to (43) belongs to $H^m_{tan}(Q_T)$, the trace of Pu_γ on Σ_T belongs to $H^m(\Sigma_T)$ and the a priori estimate*

$$\gamma||u_\gamma||^2_{H^m_{tan}(Q_T)} + ||Pu_{\gamma|\,\Sigma_T}||^2_{H^m(\Sigma_T)} \leq C_m\Big(\frac{1}{\gamma}||F_\gamma||^2_{H^m_{tan}(Q_T)} + ||G_\gamma||^2_{H^m(\Sigma_T)}\Big) \tag{45}$$

is fulfilled.

The first step to prove Theorem 15 is reducing the original problem (43) to a “stationary” boundary value problem where the time is allowed to span the whole real line and is treated, consequently, as an additional tangential variable. To make this reduction, we are going to extend the data F_γ, G_γ and the unknown u_γ of (43) to all positive and negative times, by following methods similar to those of [4, Ch.9]. In the sequel, for the sake of simplicity, we remove the subscript γ from the unknown u_γ and the data F_γ, G_γ.

Because of conditions (44), we extend F, G through $]-\infty,0]$, by setting them equal to zero for all negative times; then for $t > T$ we extend them by “reflection”, following Lions-Magenes [14, Theorem

2.2]. The extended F and G vanish for all $t > T$ sufficiently large. Let us denote by $\breve{F}$ and $\breve{G}$ the resulting extensions of F and G respectively; by construction, $\breve{F} \in H^m_{tan}(\Omega \times \mathbb{R})$ and $\breve{G} \in H^m(\partial\Omega \times \mathbb{R})$.
As we did for the data, the solution u to (43) is extended to all negative times, by setting it equal to zero. To extend u also for times $t > T$, we use assumption (E). More precisely, for every $T' > T$ we consider the problem

$$\begin{array}{l} L_\gamma u = \breve{F}_{|]0,T'[}\,, \quad \text{in } Q_{T'}\,, \\ Mu = \breve{G}_{|]0,T'[}\,, \quad \text{on } \Sigma_{T'}\,, \\ u_{|t=0} = 0\,, \quad \text{in } \Omega\,. \end{array} \tag{46}$$

By assumption (E), (46) admits a unique solution $u_{T'} \in C([0,T'];L^2(\Omega))$, such that $Pu_{T'\,|\Sigma_{T'}} \in L^2(\Sigma_{T'})$ and the energy estimate

$$\begin{aligned} &||u_{T'}(T')||^2_{L^2(\Omega)} + \gamma||u_{T'}||^2_{L^2(Q_{T'})} + ||Pu_{T'\,|\,\Sigma_{T'}}||^2_{L^2(\Sigma_{T'})} \\ &\le C'\left(\frac{1}{\gamma}||\breve{F}_{|]0,T'[}||^2_{L^2(Q_{T'})} + ||\breve{G}_{|]0,T'[}||^2_{L^2(\Sigma_{T'})}\right) \end{aligned} \tag{47}$$

is satisfied for all $\gamma \ge \gamma'$ and some constants $\gamma' \ge 1$ and $C' > 0$ depending only on T' (and the norms $||A_i||_{Lip(Q_{T'})}, ||B||_{L^\infty(Q_{T'})}$). From the uniqueness of the L^2 solution, we infer that for arbitrary $T'' > T' \ge T$ we have $u_{T''} \equiv u_{T'}$ ($u_T := u$) over $]0,T'[$; therefore, we may prolonge u beyond T, by setting it equal to the unique solution of (46) over $]0,T'[$ for all $T' > T$. Thus we define

$$\breve{u}(t) := \begin{cases} u_{T'}(t)\,, \quad \forall\, t \in]0,T'[\,,\ \forall T' > T\,, \\ 0\,, \quad \forall\, t < 0\,. \end{cases} \tag{48}$$

By construction, we have that $\breve{u}$ solves the boundary-value problem (BVP)

$$\begin{array}{l} L_\gamma u = \breve{F}\,, \quad \text{in } Q\,, \\ Mu = \breve{G}\,, \quad \text{on } \Sigma\,, \end{array} \tag{49}$$

where, hereafter, we will make use of the notations $Q := \Omega \times \mathbb{R}_t$ and $\Sigma := \partial\Omega \times \mathbb{R}_t$. In (49), the time t is involved with the same role of the tangential space variables, as it spans the whole real line $\mathbb{R}$. Since $\breve{u}$, $\breve{F}$, $\breve{G}$ are all identically zero for negative times, we can take arbitrary smooth extensions of the coefficients of the differential operator L and the boundary operator M (which are originally defined on Q_∞ and Σ_∞) on Q and Σ respectively, with the only care to preserve $rank\, A_\nu(x,t) = r$, $rank\, M(x,t) = d$ and $\ker A_\nu(x,t) \subset \ker M(x,t)$, for all $t < 0$. Let us fix such extensions once and for all; they will be still denoted by S_0, A_i, B, M. Therefore, (49) is now a stationary problem posed in Q, with data $\breve{F} \in H^m_{\tan}(Q)$, $\breve{G} \in H^m(\Sigma)$. Using the estimate (47), which holds for all $T' > T$, and noticing that $\breve{F}$, $\breve{G}$ vanish identically for large $t > 0$ (the same being true for $\breve{u}$, due to the finite speed of propagation), we derive that $\breve{u}$ enjoys the following estimate

$$\gamma||\breve{u}||^2_{L^2(Q)} + ||P\breve{u}_{|\,\Sigma}||^2_{L^2(\Sigma)} \le \breve{C}\left(\frac{1}{\gamma}||\breve{F}||^2_{L^2(Q)} + ||\breve{G}||^2_{L^2(\Sigma)}\right)\,, \tag{50}$$

for all $\gamma \ge \breve{\gamma}$, and suitable constants $\breve{\gamma} \ge 1$, $\breve{C} > 0$.
The proof of Theorem 15 will be derived as a consequence of the tangential regularity of the BVP (49). Thus we concentrate from now on this problem.
In the sequel, for the sake of simplicity, we remove the superscript from the unknown $\breve{u}$ and the data $\breve{F}$, $\breve{G}$ of (49). For the sake of convenience, let us also set $\mathcal{N}(x,t) := \ker A_\nu(x,t)$, $\mathcal{M}(x,t) := \ker M(x,t)$. By assumption (C) we have $\mathcal{N}(x,t) \subset \mathcal{M}(x,t)$ for $(x,t) \in \Sigma_\infty$ (and also for all negative times). Recall that $d = N - \dim\mathcal{M} = \operatorname{rank}\mathcal{M}$ is equal to the number of negative eigenvalues of A_ν and $r = \operatorname{rank} A_\nu = N - \dim\mathcal{N}$; then $d \le r < N$.
The next step is to move from the BVP (49) to a similar BVP posed in the $(n+1)-$dimensional positive half-space $\mathbb{R}^{n+1}_+ := \{(x_1, x', t) : x_1 > 0,\, (x',t) \in \mathbb{R}^n\}$. In order to make the wanted reduction in a proper way, we first need some technical lemmata.

Lemma 16. *Assume* (A)-(D). *For each $\overline{x} \in \partial\Omega$ there exists a neighborhood $\mathcal{U}$ of $\overline{x}$ in $\mathbb{R}^n$ and an $N \times N$ unitary matrix-valued function $T(x,t) \in C^\infty((\mathcal{U} \cap \overline{\Omega}) \times \mathbb{R})$ such that, for all $(x,t) \in (\mathcal{U} \cap \partial\Omega) \times \mathbb{R}$,*

$u \in \mathcal{M}(x,t)$ is equivalent to $T(x,t)u \in \hat{\mathcal{M}} = \{w \in \mathbb{R}^N : w_1 = \cdots = w_d = 0\}$,and $u \in \mathcal{N}(x,t)$ is equivalent to $T(x,t)u \in \hat{\mathcal{N}} = \{w \in \mathbb{R}^N : w_1 = \cdots = w_r = 0\}$.

Proof. See [37, Lemma 1]. Here the assumption (C), $M \in C^\infty$, is used in order to find a local smoothly varying orthonormal basis of $\mathcal{M}(x,t)^\perp \subset \mathcal{N}(x,t)^\perp$. The assumption (D), $P \in C^\infty$, is used in order to complete this basis and find a smoothly varying orthonormal basis of $\mathcal{N}(x,t)^\perp$; it is used also in order to find a smooth orthonormal basis of $\mathcal{N}(x,t)$. Thus one has a locally defined smooth orthonormal basis of $\mathbb{R}^N$ which is used for the construction of the unitary matrix $T(x,t)$. □

Lemma 17. *Assume* (A)-(D). *For each $\overline{x} \in \partial\Omega$, let $T(x,t)$ be the unitary matrix defined in $(\mathcal{U} \cap \overline{\Omega}) \times \mathbb{R}$, by Lemma 16. Let us define the matrix $\widehat{A}_\nu(x,t) := T(x,t)S_0(x,t)A_\nu(x,t)T(x,t)^*$, $(x,t) \in (\mathcal{U} \cap \overline{\Omega}) \times \mathbb{R}$. Then $\widehat{A}_\nu$ is symmetric and may be written in the following block form*

$$\widehat{A}_\nu(x,t) = \begin{pmatrix} \widehat{A}_1^{I,I} & \widehat{A}_1^{I,II} \\ \\ \widehat{A}_1^{II,I} & \widehat{A}_1^{II,II} \end{pmatrix}, \ (x,t) \in (\mathcal{U} \cap \overline{\Omega}) \times \mathbb{R}, \tag{51}$$

where $\widehat{A}_1^{I,I}\,\widehat{A}_1^{I,II}, \widehat{A}_1^{II,I}\,\widehat{A}_1^{II,II}$ are respectively $r \times r$, $r \times (N-r)$, $(N-r) \times r$, $(N-r) \times (N-r)$ sub-matrices. Moreover, $\widehat{A}_1^{I,I}(x,t)$ is invertible on $(\mathcal{U} \cap \overline{\Omega}) \times \mathbb{R}$ and we have

$$\widehat{A}_1^{I,II} = 0, \quad \widehat{A}_1^{II,I} = 0, \quad \widehat{A}_1^{II,II} = 0, \quad \text{in } (\mathcal{U} \cap \partial\Omega) \times \mathbb{R}. \tag{52}$$

Proof. See [29, Lemma 3.2]. □

Correspondingly we will decompose u as $u = (u^I, u^{II})$. After the change of dependent variables associated to the unitary matrix $T(x,t)$ we have $Pu = (u^I, 0)$.

To make the announced reduction into a problem in $\mathbb{R}^{n+1}_+$, we follow a standard localization procedure of the problem (49) near the boundary of the spatial domain Ω. We take a covering $\{U_j\}_{j=0}^l$ of Ω and a partition of unity $\{\psi_j\}_{j=0}^l$ subordinate to this covering, as in Section 2. We assume that each patch $U_j, j = 1, \ldots, l$, is so small that one can find the unitary matrix $T_j(x,t)$ there defined, as above. Let us even set $T_0 := I_n$.

Given the solution u to (49), for each $j = 0, \ldots, l$ we denote $u^j(x,t) := T_j(x,t)\psi_j(x)u(x,t)$. For each $j = 1, \ldots, l$, in local coordinates as in Section 2, u^j solves in the half-space $\mathbb{R}^{n+1}_+$ a BVP of the form

$$\begin{array}{ll} (\gamma A_0^j + L^j)u^j = F^j + K^j u, & \text{in } \mathbb{R}^{n+1}_+\,, \\ \widetilde{M}u^j = G^j, & \text{on } \mathbb{R}^n\,, \end{array} \tag{53}$$

with

$$L^j := A_0^j(x,t)\partial_t + \textstyle\sum_{i=1}^n A_i^j(x,t)\partial_i + B^j(x,t), \qquad \widetilde{M} := (I_d\,,\,0)\,. \tag{54}$$

Here we have denoted

$$\begin{aligned} & A_0^j = T_j S_0 T_j^*, \quad A_i^j = T_j S_0 A_i T_j^*, \\ & B^j = T_j S_0 B T_j^* + T_j S_0 \partial_t T_j^* + \sum_k T_j S_0 A_k \partial_k T_j^*, \\ & K^j = \sum_k T_j S_0 A_k \partial_k \psi_j, \quad F^j = T_j \psi_j S_0 F, \quad G^j = \psi_j G. \end{aligned} \tag{55}$$

The boundary matrix $-A_1^j$ has the block form as in (51), (52).
Note that the solution u^j to (53), as well as the right-hand side $F^j + K^j u$ are compactly supported on $\{x_1 \geq 0\,,\ |x| < 1\} \times [0, +\infty[$, whereas the boundary data G^j's are compactly supported on $\{x_1 = 0\,,\ |x| < 1\} \times [0, +\infty[$. After multiplicating by a suitable cut-off function, we may assume that A_i^j, B^j, $i = 0, \ldots, n$, have compact support as well, namely that A_i^j, B^j are in $C^\infty_{(0)}(\mathbb{R}^{n+1}_+)$, as required by Lemmata 8-10.
As a next step we will prove that we can attach to the problem (53) a local counterpart of the global estimate (50) associated to the stationary problem (49). More precisely, one can prove the following

Lemma 18. *For all $j = 1, \dots, l$, let L^j be the local differential operator* (54)*, written in a system of local coordinates mapping $U_j \cap \overline{\Omega}$ onto the unit half ball $\mathbb{B} := \{x_1 \geq 0\,,\ |x| < 1\}$ (and $U_j \cap \partial\Omega$ onto $\mathbb{D} := \{x_1 = 0\,,\ |x| < 1\}$). Then there exist constants $C_j > 0$ and $\gamma_j \geq 1$ such that for all $\varphi \in L^2(\mathbb{B} \times [0, +\infty[)$ such that $L^j\varphi \in L^2(\mathbb{B} \times [0, +\infty[)$ and $\gamma \geq \gamma_j$ one has*

$$\gamma ||\varphi||^2_{L^2(\mathbb{R}^{n+1}_+)} + ||\varphi^I_{|x_1=0}||^2_{L^2(\mathbb{R}^n)} \leq C_j(\frac{1}{\gamma}||(\gamma A^j_0 + L^j)\varphi||^2_{L^2(\mathbb{R}^{n+1}_+)} + ||\widetilde{M}\varphi_{|x_1=0}||^2_{L^2(\mathbb{R}^n)})\,. \tag{56}$$

Proof. By density, we reduce to prove the estimate above only for $\varphi \in C^\infty_{(0)}(\mathbb{B} \times [0, +\infty[)$. We follow here the same lines of [6]. For each j, let T_j be the unitary matrix of Lemma 16; let us denote

$$\chi_j : U_j \cap \overline{\Omega} \to \mathbb{B},$$

the local change of coordinates associated to the coordinate patch U_j, such that $\chi_j(U_j \cap \partial\Omega) = \mathbb{D}$. Agreeing with the notations used in [6], we set

$$\chi_{j\,*} v := v \circ \chi_j^{-1}\,, \quad \chi_j^* w = w \circ \chi_j\,,$$

whenever v and w are functions defined respectively on $U_j \cap \overline{\Omega}$ and $\mathbb{B}$. According to the conventions before, in local coordinates χ_j the problem (53) reads as

$$\begin{aligned} &\chi_{j\,*}(\gamma A^j_0 + L^j)\chi_j^*(\chi_{j\,*}u^j) = \chi_{j\,*}F^j + \chi_{j\,*}K^j u, && \text{in } \mathbb{R}^{n+1}_+\,, \\ &\widetilde{M}\chi_{j\,*}u^j = \chi_{j\,*}G^j, && \text{on } \mathbb{R}^n\,. \end{aligned} \tag{57}$$

Using (55) and that T_j is a unitary matrix, we compute that

$$S_0^{-1}T_j^*(\gamma A^j_0 + L^j) = L_\gamma T_j^*\,, \tag{58}$$

on $(U_j \cap \overline{\Omega}) \times \mathbb{R}$. For all $\varphi \in C^\infty_{(0)}(\mathbb{B} \times [0, +\infty[)$ we get

$$\begin{aligned} &\gamma||\varphi||^2_{L^2(\mathbb{R}^{n+1}_+)} + ||\varphi^I_{|x_1=0}||^2_{L^2(\mathbb{R}^n)} \leq C_j\left(\gamma||\chi_j^*\varphi||^2_{L^2(Q)} + ||\chi_j^*\varphi^I_{|\Sigma}||^2_{L^2(\Sigma)}\right) \\ &= C_j\left(\gamma||T_j^*\chi_j^*\varphi||^2_{L^2(Q)} + ||P(T_j^*\chi_j^*\varphi)_{|\Sigma}||^2_{L^2(\Sigma)}\right)\,. \end{aligned} \tag{59}$$

Then, applying estimate (50) with $T_j^*\chi_j^*\varphi$ instead of $\breve{u}$ and using (58), we get

$$\begin{aligned} &\gamma||T_j^*\chi_j^*\varphi||^2_{L^2(Q)} + ||P(T_j^*\chi_j^*\varphi)_{|\Sigma}||^2_{L^2(\Sigma)} \leq C_0(\frac{1}{\gamma}||L_\gamma(T_j^*\chi_j^*\varphi)||^2_{L^2(Q)} + ||MT_j^*\chi_j^*\varphi_{|\Sigma}||^2_{L^2(\Sigma)}) \\ &= C_0(\frac{1}{\gamma}||S_0^{-1}T_j^*(\gamma A^j_0 + L^j)\chi_j^*\varphi||^2_{L^2(Q)} + ||\widetilde{M}\chi_j^*\varphi_{|\Sigma}||^2_{L^2(\Sigma)}) \\ &\leq \widetilde{C}_0(\frac{1}{\gamma}||T_j^*(\gamma A^j_0 + L^j)\chi_j^*\varphi||^2_{L^2(Q)} + ||\widetilde{M}\chi_j^*\varphi_{|\Sigma}||^2_{L^2(\Sigma)}) \\ &= \widetilde{C}_0(\frac{1}{\gamma}||(\gamma A^j_0 + L^j)\chi_j^*\varphi||^2_{L^2(Q)} + ||\widetilde{M}\chi_j^*\varphi_{|\Sigma}||^2_{L^2(\Sigma)}) \\ &\leq \widetilde{C}_j(\frac{1}{\gamma}||\chi_{j\,*}(\gamma A^j_0 + L^j)\chi_j^*\varphi||^2_{L^2(\mathbb{R}^{n+1}_+)} + ||\widetilde{M}\varphi_{|x_1=0}||^2_{L^2(\mathbb{R}^n)})\,. \end{aligned} \tag{60}$$

We observe that the first term in the last row of (60) is the first term of the right-hand side of (56) written with different notation. From (59), (60) we get (56). ◻

In the sequel, for the sake of simplicity, we will remove all the indices and write v instead of u^j, F instead of $F^j + K^j u$, and so on. From now on $M = (I_d\,, 0)$. Moreover it will be convenient to recover the notations $x_{n+1} := t$ and $x := (x_1, x', x_{n+1})$.

For $\varepsilon \in]0, 1[$, let us introduce the mollified solution $J_\varepsilon v$. Then $J_\varepsilon v$ clearly satisfies the following equation

$$(\gamma A_0 + L)J_\varepsilon v = J_\varepsilon F + [\gamma A_0 + L, J_\varepsilon]v\,, \ \text{ in } \mathbb{R}^{n+1}_+\,, \tag{61}$$

As regards the boundary conditions we have the following lemmata.

Lemma 19. *Let v be a function in $L^2(\mathbb{R}^{n+1}_+)$ such that $Lv \in L^2(\mathbb{R}^{n+1}_+)$; then*

$$J_\varepsilon v^I_{|\{x_1=0\}} = v^I_{|\{x_1=0\}} * \widetilde{\chi}_\varepsilon. \tag{62}$$

Proof. Let us denote by $C^1_{(0)}(\mathbb{R}^{n+1}_+)$ the restriction onto $\mathbb{R}^{n+1}_+$ of functions in $C^1(\mathbb{R}^{n+1})$ with compact support. In view of [22, Theorems 7 and 8], there exists a sequence $\{v_\nu\}_{\nu\geq 0}$ of functions $v_\nu \in C^1_{(0)}(\mathbb{R}^{n+1}_+)$ such that

$$v_\nu \to v\,, \quad \text{in } L^2(\mathbb{R}^{n+1}_+)\,, \tag{63}$$

$$Lv_\nu \to Lv\,, \quad \text{in } L^2(\mathbb{R}^{n+1}_+)\,, \tag{64}$$

$$A_1 v_{\nu\,|\,\{x_1=0\}} \to A_1 v_{|\,\{x_1=0\}}\,, \quad \text{in } H^{-1/2}(\mathbb{R}^n)\,. \tag{65}$$

From (51) and (52), (65) readily gives

$$v^I_{\nu\,|\,\{x_1=0\}} \to v^I_{|\,\{x_1=0\}}\,, \quad \text{in } H^{-1/2}(\mathbb{R}^n)\,. \tag{66}$$

Property (24) of the mollifiers J_ε and (63) imply

$$J_\varepsilon v_\nu \to J_\varepsilon v\,, \quad \text{in } L^2(\mathbb{R}^{n+1}_+)\,. \tag{67}$$

Writing

$$LJ_\varepsilon v_\nu = J_\varepsilon L v_\nu + [L, J_\varepsilon] v_\nu\,,$$

and using (24), (64) and the estimate

$$||[L, J_\varepsilon]v||_{L^2(\mathbb{R}^{n+1}_+)} \leq C(\frac{1}{\varepsilon}||v||_{L^2(\mathbb{R}^{n+1}_+)} + ||Lv||_{L^2(\mathbb{R}^{n+1}_+)}) \tag{68}$$

(whose proof follows the same lines of similar calculations made below and is therefore omitted), we obtain

$$LJ_\varepsilon v_\nu \to LJ_\varepsilon v\,, \quad \text{in } L^2(\mathbb{R}^{n+1}_+)\,. \tag{69}$$

By [22, Theorem 1], (67), (69),(51) and (52), we infer

$$J_\varepsilon v^I_{\nu\,|\,\{x_1=0\}} \to J_\varepsilon v^I_{|\,\{x_1=0\}}\,, \quad \text{in } H^{-1/2}(\mathbb{R}^n)\,. \tag{70}$$

Since v_ν are smooth functions, by Fubini's theorem one computes for all $\nu \geq 0$

$$J_\varepsilon v^I_{\nu\,|\,\{x_1=0\}}(x') = \int_{\mathbb{R}^{n+1}} v^I_\nu(0, x'-y') e^{-\frac{y_1}{2}} \chi_\varepsilon(y_1, y') dy_1 dy'$$

$$= \int_{\mathbb{R}^n} \widetilde{\chi}_\varepsilon(y') v^I_\nu(0, x'-y') dy' = v^I_{\nu\,|\,\{x_1=0\}} * \widetilde{\chi}_\varepsilon(x').$$

Then (62) is obtained by letting $\nu \to \infty$ in the identity above and using (70) and

$$v^I_{\nu\,|\,\{x_1=0\}} * \widetilde{\chi}_\varepsilon \to v^I_{|\,\{x_1=0\}} * \widetilde{\chi}_\varepsilon \quad \text{in } H^{-1/2}(\mathbb{R}^n)\,,$$

which follows from (66) and the known properties of the convolution. □

Lemma 20. *Let $v \in L^2(\mathbb{R}^{n+1}_+)$ be a solution to* (53)*. Then*

$$MJ_\varepsilon v = G * \widetilde{\chi}_\varepsilon\,, \ \text{on } \{x_1 = 0\} \times \mathbb{R}^n_{x',t}. \tag{71}$$

Proof. Recalling that $Pv = (v^I, 0)$, from (62) we derive

$$\begin{aligned} MJ_\varepsilon v_{|\,\{x_1=0\}} &= MPJ_\varepsilon v|_{\{x_1=0\}} = M(Pv_{|\,\{x_1=0\}} * \widetilde{\chi}_\varepsilon) \\ &= MPv_{|\,\{x_1=0\}} * \widetilde{\chi}_\varepsilon = Mv_{|\,\{x_1=0\}} * \widetilde{\chi}_\varepsilon = G * \widetilde{\chi}_\varepsilon\,. \end{aligned}$$

□

We continue the proof of Theorem 15.

Observe in particular that $\operatorname{supp} J_\varepsilon v \subset \mathbb{B} \times [0, +\infty[$ and $J_\varepsilon v = 0$ for all $t > 0$ sufficiently large and ε small, because of (39). Thus we can apply estimate (56) of Lemma 18, in order to obtain the L^2 energy estimate

$$\gamma||J_\varepsilon v||^2_{L^2(\mathbb{R}^{n+1}_+)} + ||J_\varepsilon v^I_{|\,\{x_1=0\}}||^2_{L^2(\mathbb{R}^n)} \leq C_0 \left(\frac{1}{\gamma}||J_\varepsilon F + [\gamma A_0 + L, J_\varepsilon]v||^2_{L^2(\mathbb{R}^{n+1}_+)} + ||G * \widetilde{\chi}_\varepsilon||^2_{L^2(\mathbb{R}^n)} \right), \tag{72}$$

for all $\gamma \geq \gamma_0$. From Lemma 10 we have

$$\begin{aligned}&\int_0^1 ||[A_0, J_\varepsilon]v||^2_{L^2(\mathbb{R}^{n+1}_+)}\varepsilon^{-2k}\left(1+\tfrac{\delta^2}{\varepsilon^2}\right)^{-1}\tfrac{d\varepsilon}{\varepsilon}+\\&+\int_0^1 ||[B, J_\varepsilon]v||^2_{L^2(\mathbb{R}^{n+1}_+)}\varepsilon^{-2k}\left(1+\tfrac{\delta^2}{\varepsilon^2}\right)^{-1}\tfrac{d\varepsilon}{\varepsilon}\leq C||v||^2_{\mathbb{R}^{n+1}_+,k-2,tan,\delta}\,,\end{aligned}\tag{73}$$

and, for $j = 2, \cdots, n$,

$$\int_0^1 ||[A_jZ_j, J_\varepsilon]v||^2_{L^2(\mathbb{R}^{n+1}_+)}\varepsilon^{-2k}\left(1+\tfrac{\delta^2}{\varepsilon^2}\right)^{-1}\tfrac{d\varepsilon}{\varepsilon}\leq C||v||^2_{\mathbb{R}^{n+1}_+,k-1,tan,\delta},\tag{74}$$

for all $k = 1, \ldots, m$. As regards the commutator $[A_1\partial_1, J_\varepsilon]v$ we prove the following lemma.

Lemma 21. *For $k = 1, \ldots, m$, let $v \in H^{k-1}_{tan}(\mathbb{R}^{n+1}_+)$ be a solution to* (53)*. Then there exists a constant $C > 0$ such that the following estimate*

$$\begin{aligned}&\int_0^1 ||[A_1\partial_1, J_\varepsilon]v||^2_{L^2(\mathbb{R}^{n+1}_+)}\varepsilon^{-2k}\left(1+\frac{\delta^2}{\varepsilon^2}\right)^{-1}\frac{d\varepsilon}{\varepsilon}\\&\leq C\left(\int_0^1 ||J_\varepsilon v||^2_{L^2(\mathbb{R}^{n+1}_+)}\varepsilon^{-2k}\left(1+\frac{\delta^2}{\varepsilon^2}\right)^{-1}\frac{d\varepsilon}{\varepsilon}+\gamma^2||v||^2_{\mathbb{R}^{n+1}_+,k-1,tan}+||F||^2_{\mathbb{R}^{n+1}_+,k-2,tan,\delta}\right),\end{aligned}\tag{75}$$

holds for all $0 < \delta \leq 1$ and for all γ large enough.

Proof. Recalling (51), (52), let us decompose A_1 as

$$A_1 = A_1^1 + A_1^2 := \begin{pmatrix} A_1^{I,I} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & A_1^{I,II} \\ A_1^{II,I} & A_1^{II,II} \end{pmatrix}.\tag{76}$$

Since A_1^2 vanishes at $\{x_1 = 0\}$ we can write it as

$$A_1^2(x,t) = H(x,t)x_1, \quad \text{with } H(x_1,x',t) := \tfrac{1}{x_1}\int_0^{x_1}\partial_1A_1^2(y,x',t)dy,\tag{77}$$

and $H \in C^\infty_{(0)}(\mathbb{R}^{n+1}_+)$. This gives $A_1^2\partial_1 = HZ_1$ so that, again from Lemma 10, we have

$$\int_0^1 ||[A_1^2\partial_1, J_\varepsilon]v||^2_{L^2(\mathbb{R}^{n+1}_+)}\varepsilon^{-2k}\left(1+\tfrac{\delta^2}{\varepsilon^2}\right)^{-1}\tfrac{d\varepsilon}{\varepsilon}\leq C||v||^2_{\mathbb{R}^{n+1}_+,k-1,tan,\delta},\tag{78}$$

for all $k = 1, \ldots, m$.

Disregarding the different size of matrices and vectors, we write for the sake of simplicity $[A_1^1\partial_1, J_\varepsilon]v = [A_1^{I,I}\partial_1, J_\varepsilon]v^I$, where it is understood that the matrix $A_1^{I,I}$ is everywhere invertible.

Using properties (16), (23), (26), we can derive

$$\begin{aligned}&\left([A_1^{I,I}\partial_1, J_\varepsilon]v^I\right)^\sharp = A_1^{I,I\natural}(\partial_1J_\varepsilon v^I)^\sharp - (A_1^{I,I\natural}(\partial_1v^I)^\sharp)*\chi_\varepsilon =\\&= A_1^{I,I\natural}e^{-x_1}\left(Z_1J_\varepsilon v^I\right)^\sharp - \left(A_1^{I,I\natural}e^{-x_1}(Z_1v^I)^\sharp\right)*\chi_\varepsilon =\\&= A_1^{I,I\natural}e^{-x_1}\left(J_\varepsilon Z_1 v^I\right)^\sharp - \left(A_1^{I,I\natural}e^{-x_1}(Z_1v^I)^\sharp\right)*\chi_\varepsilon =\\&= A_1^{I,I\natural}e^{-x_1}\left(Z_1 v^I\right)^\sharp*\chi_\varepsilon - \left(A_1^{I,I\natural}e^{-x_1}(Z_1v^I)^\sharp\right)*\chi_\varepsilon =\\&= \int_{\mathbb{R}^{n+1}}\left[A_1^{I,I\natural}(x)e^{-x_1} - A_1^{I,I\natural}(x-y)e^{-(x_1-y_1)}\right](Z_1v^I)^\sharp(x-y)\chi_\varepsilon(y)dy\\&= \int_{\mathbb{R}^{n+1}}\left[A_1^{I,I\natural}(x)e^{-y_1} - A_1^{I,I\natural}(x-y)\right](\partial_1v^I)^\sharp(x-y)\chi_\varepsilon(y)dy.\end{aligned}\tag{79}$$

Because of the invertibility of $A_1^{I,I}$, from (53) we have (where $A_{n+1}\partial_{n+1} = A_0\partial_t$)

$$\partial_1 v^I = (A_1^{I,I})^{-1}\left(-A_1^{I,II}\partial_1 v^{II} + \left(F - \gamma A_0 v - \sum_{i=2}^{n+1} A_i\partial_i v - Bv\right)^I\right).$$

Inserting the right-hand side of this equality in (79) gives different terms that we now analyse one by one. Let us start with

$$J_1 := -\int_{\mathbb{R}^{n+1}} \left[A_1^{I,I\natural}(x)e^{-y_1} - A_1^{I,I\natural}(x-y)\right]\left((A_1^{I,I})^{-1}A_1^{I,II}\partial_1 v^{II}\right)^\sharp (x-y)\chi_\varepsilon(y)dy.$$

Since $A_1^{I,II}$ vanishes at $\{x_1 = 0\}$, we can write

$$-(A_1^{I,I})^{-1}A_1^{I,II}\partial_1 v^{II} = H_1 Z_1 v,$$

with $H_1 \in C^\infty_{(0)}(\mathbb{R}^{n+1}_+)$ defined as in (77). Then we infer

$$J_1 = \int_{\mathbb{R}^{n+1}} \left[A_1^{I,I\natural}(x)e^{-y_1} - A_1^{I,I\natural}(x-y)\right] H_1^\natural(x-y)\left(\partial_1 v^\sharp(x-y) - \tfrac{1}{2}v^\sharp(x-y)\right)\chi_\varepsilon(y)dy,$$

where we have used $(H_1Z_1v)^\sharp = H_1^\natural(\partial_1 v^\sharp - \frac{1}{2}v^\sharp)$. We notice that we can write

$$\left[A_1^{I,I\natural}(x)e^{-y_1} - A_1^{I,I\natural}(x-y)\right] H_1^\natural(x-y) = b_1(x,y)\cdot y,$$

with $b_1(x,y) \in \mathcal{B}^\infty(\mathbb{R}^{n+1}\times\mathbb{R}^{n+1})$. Integrating by parts yields

$$J_1 = \int_{\mathbb{R}^{n+1}} \left(\tfrac{\partial}{\partial y_1}(b_1(x,y)\cdot y) - \tfrac{1}{2}\, b_1(x,y)\cdot y\right) v^\sharp(x-y)\chi_\varepsilon(y)dy$$

$$+\tfrac{1}{\varepsilon}\int_{\mathbb{R}^{n+1}} b_1(x,y)\cdot y\, v^\sharp(x-y)(\partial_1\chi)_\varepsilon(y)dy.$$

Thus J_1 can be written as a sum of terms of the form (31) and (32). By applying Lemma 9 and (36), as in the proof of the second inequality of Lemma 10, we infer

$$\int_0^1 ||J_1||^2_{L^2(\mathbb{R}^{n+1}_+)}\varepsilon^{-2k}\left(1 + \tfrac{\delta^2}{\varepsilon^2}\right)^{-1}\tfrac{d\varepsilon}{\varepsilon} \le C||v||^2_{\mathbb{R}^{n+1}_+,k-1,tan,\delta}\,. \tag{80}$$

Let us set

$$J_2 := \int_{\mathbb{R}^{n+1}} \left[A_1^{I,I\natural}(x)e^{-y_1} - A_1^{I,I\natural}(x-y)\right]\left((A_1^{I,I})^{-1}(F - \gamma A_0 v - Bv)^I\right)^\sharp (x-y)\chi_\varepsilon(y)dy.$$

We notice that we can write

$$\left[A_1^{I,I\natural}(x)e^{-y_1} - A_1^{I,I\natural}(x-y)\right]\left((A_1^{I,I})^{-1}\right)^\natural (x-y) = b_2(x,y)\cdot y,$$

with $b_2(x,y) \in \mathcal{B}^\infty(\mathbb{R}^{n+1}\times\mathbb{R}^{n+1})$. Thus J_2 can be written as a sum of terms of the form (30) in F and v. By (36) we infer

$$\int_0^1 ||J_2||^2_{L^2(\mathbb{R}^{n+1}_+)}\varepsilon^{-2k}\left(1 + \tfrac{\delta^2}{\varepsilon^2}\right)^{-1}\tfrac{d\varepsilon}{\varepsilon} \le C||F||^2_{\mathbb{R}^{n+1}_+,k-2,tan,\delta} + C\gamma^2||v||^2_{\mathbb{R}^{n+1}_+,k-2,tan,\delta}\,, \tag{81}$$

for γ large enough. Finally let us set

$$J_3 := -\sum_{i=2}^{n+1}\int_{\mathbb{R}^{n+1}} \left[A_1^{I,I\natural}(x)e^{-y_1} - A_1^{I,I\natural}(x-y)\right]\left((A_1^{I,I})^{-1}(A_i\partial_i v)^I\right)^\sharp (x-y)\chi_\varepsilon(y)dy.$$

Arguing as for (80) we infer

$$\int_0^1 ||J_3||^2_{L^2(\mathbb{R}^{n+1}_+)}\varepsilon^{-2k}\left(1 + \tfrac{\delta^2}{\varepsilon^2}\right)^{-1}\tfrac{d\varepsilon}{\varepsilon} \le C||v||^2_{\mathbb{R}^{n+1}_+,k-1,tan,\delta}\,. \tag{82}$$

Adding estimates (78), (80), (81), (82), and applying Lemma 5 and Proposition 6 eventually gives (75) for γ large enough. $\square$

From (73), (74), (75) we infer

$$\begin{aligned}&\int_0^1 ||[\gamma A_0+L,J_\varepsilon]v||^2_{L^2(\mathbb{R}^{n+1}_+)}\varepsilon^{-2k}\left(1+\frac{\delta^2}{\varepsilon^2}\right)^{-1}\frac{d\varepsilon}{\varepsilon}\\&\le C\left(\int_0^1 ||J_\varepsilon v||^2_{L^2(\mathbb{R}^{n+1}_+)}\varepsilon^{-2k}\left(1+\frac{\delta^2}{\varepsilon^2}\right)^{-1}\frac{d\varepsilon}{\varepsilon}+\gamma^2||v||^2_{\mathbb{R}^{n+1}_+,k-1,tan}+||F||^2_{\mathbb{R}^{n+1}_+,k-2,tan,\delta}\right).\end{aligned}\tag{83}$$

Because $G\in H^m(\mathbb{R}^n)$, from Proposition 12 and Lemma 13 we obtain

$$\int_0^1 ||G*\widetilde{\chi}_\varepsilon||^2_{L^2(\mathbb{R}^n)}\varepsilon^{-2k}\left(1+\frac{\delta^2}{\varepsilon^2}\right)^{-1}\frac{d\varepsilon}{\varepsilon}\le C||G||^2_{\mathbb{R}^n,k}\,,\tag{84}$$

for all $k=1,\dots,m$. From (72), (83), (84), and Proposition 6, by taking γ large enough, we obtain

$$\begin{aligned}&\gamma\int_0^1 ||J_\varepsilon v||^2_{L^2(\mathbb{R}^{n+1}_+)}\varepsilon^{-2k}\left(1+\frac{\delta^2}{\varepsilon^2}\right)^{-1}\frac{d\varepsilon}{\varepsilon}+\int_0^1 ||J_\varepsilon v^I_{|\{x_1=0\}}||^2_{L^2(\mathbb{R}^n)}\varepsilon^{-2k}\left(1+\frac{\delta^2}{\varepsilon^2}\right)^{-1}\frac{d\varepsilon}{\varepsilon}\\&\le C\left(\frac{1}{\gamma}||F||^2_{\mathbb{R}^{n+1}_+,k-1,tan,\delta}+||G||^2_{\mathbb{R}^n,k}+\gamma||v||^2_{\mathbb{R}^{n+1}_+,k-1,tan}\right),\end{aligned}$$

for all $k=1,\dots,m$. Adding to both sides $\gamma||v||^2_{\mathbb{R}^{n+1}_+,k-1,tan}+||v^I_{|\{x_1=0\}}||^2_{\mathbb{R}^n,k-1}$ and taking account of Proposition 6, Lemma 19 and Proposition 12, yields

$$\begin{aligned}&\gamma||v||^2_{\mathbb{R}^{n+1}_+,k-1,tan,\delta}+||v^I_{|\{x_1=0\}}||^2_{\mathbb{R}^n,k-1,\delta}\\&\le C\left(\frac{1}{\gamma}||F||^2_{\mathbb{R}^{n+1}_+,k-1,tan,\delta}+||G||^2_{\mathbb{R}^n,k}+\gamma||v||^2_{\mathbb{R}^{n+1}_+,k-1,tan}+||v^I_{|\{x_1=0\}}||^2_{\mathbb{R}^n,k-1}\right),\end{aligned}\tag{85}$$

for all $k=1,\dots,m$ and γ large enough. By Lemma 18, v also enjoys the L^2 energy estimate

$$\gamma||v||^2_{L^2(\mathbb{R}^{n+1}_+)}+||v^I_{|\{x_1=0\}}||^2_{L^2(\mathbb{R}^n)}\le C_0\left(\frac{1}{\gamma}||F||^2_{L^2(\mathbb{R}^{n+1}_+)}+||G||^2_{L^2(\mathbb{R}^n)}\right),\tag{86}$$

for all $\gamma\ge\gamma_0$. Using (85) recursively for $k=1,\dots,m$, and taking account of (86) and Lemma 5 gives

$$\gamma||v||^2_{\mathbb{R}^{n+1}_+,m-1,tan,\delta}+||v^I_{|\{x_1=0\}}||^2_{\mathbb{R}^n,m-1,\delta}\le C\Big(\frac{1}{\gamma}||F||^2_{\mathbb{R}^{n+1}_+,m-1,tan,\delta}+||G||^2_{\mathbb{R}^n,m}\Big),\tag{87}$$

for all γ large enough and for all $0<\delta\le1$. Recovering the original notation of (53), estimate (87) reads

$$\begin{aligned}&\gamma||u^j||^2_{\mathbb{R}^{n+1}_+,m-1,tan,\delta}+||u^{jI}_{|\{x_1=0\}}||^2_{\mathbb{R}^n,m-1,\delta}\\&\le C\Big(\frac{1}{\gamma}||F^j+K^ju||^2_{\mathbb{R}^{n+1}_+,m-1,tan,\delta}+||G^j||^2_{\mathbb{R}^n,m}\Big)\\&\le C\Big(\frac{1}{\gamma}||F^j||^2_{\mathbb{R}^{n+1}_+,m,tan}+\frac{1}{\gamma}\sum_{j=0}^{l}||u^j||^2_{\mathbb{R}^{n+1}_+,m-1,tan,\delta}+||G^j||^2_{\mathbb{R}^n,m}\Big),\end{aligned}\tag{88}$$

for all γ large enough, for all $0<\delta\le1$ and each $j=1,\dots,l$.
By simpler calculations, an estimate analogous to (88) (without boundary terms) can be derived for $u^0=\psi_0u$, where tangential norms in $\mathbb{R}^{n+1}_+$ are replaced by usual Sobolev norms in $\mathbb{R}^{n+1}$. Summing over all $j=0,\dots,l$ the previous estimates and taking γ large enough, we derive

$$\begin{aligned}&\gamma||u^0||^2_{\mathbb{R}^{n+1},m-1,\delta}+\gamma\sum_{j=1}^{l}||u^j||^2_{\mathbb{R}^{n+1}_+,m-1,tan,\delta}+\sum_{j=1}^{l}||u^{jI}_{|\{x_1=0\}}||^2_{\mathbb{R}^n,m-1,\delta}\\&\le C\Big(\frac{1}{\gamma}||F^0||^2_{\mathbb{R}^{n+1},m}+\frac{1}{\gamma}\sum_{j=1}^{l}||F^j||^2_{\mathbb{R}^{n+1}_+,m,tan}+\sum_{j=1}^{l}||G^j||^2_{\mathbb{R}^n,m}\Big)=C\Big(\frac{1}{\gamma}||F||^2_{H^m_{\tan}(Q)}+||G||^2_{H^m(\Sigma)}\Big).\end{aligned}\tag{89}$$

The above inequality shows that $u\in H^m_{\tan}(Q)$ and $Pu\in H^m(\Sigma)$ (cf. Lemma 5 and [13]). Passing to the limit as $\delta\downarrow0$ in (89) gives the apriori estimate

$$\gamma||u||^2_{H^m_{tan}(Q)}+||Pu|\,\Sigma||^2_{H^m(\Sigma)}\le C\Big(\frac{1}{\gamma}||F||^2_{H^m_{tan}(Q)}+||G||^2_{H^m(\Sigma)}\Big),\tag{90}$$

for all γ large enough. This concludes the proof of the tangential regularity of the solution u of the BVP (49).

Recalling that this solution u is the extension of the solution u_γ of the original IBVP (43), from the tangential regularity of u we can now derive the tangential regularity of u_γ, namely that $u_\gamma \in H^m_{\tan}(Q_T)$ and $Pu_\gamma \in H^m(\Sigma_T)$. To get the energy estimate (45), we observe that extended data $\breve{F}$ and $\breve{G}$ are defined in such a way that

$$||\breve{F}||_{H^m_{\tan}(Q)} \le C||F_\gamma||_{H^m_{\tan}(Q_T)}, \quad ||\breve{G}||_{H^m(\Sigma)} \le C||G_\gamma||_{H^m(\Sigma_T)} ,$$

with positive constant C independent of F_γ, G_γ and γ. Hence, from the energy estimate (90), we infer

$$\begin{aligned}&\gamma||u_\gamma||^2_{H^m_{\tan}(Q_T)} + ||Pu_{\gamma\,|\Sigma_T}||^2_{H^m(\Sigma_T)} \le \gamma||u||^2_{H^m_{tan}(Q)} + ||Pu_{|\,\Sigma}||^2_{H^m(\Sigma)}\\ &\le C\Big(\tfrac{1}{\gamma}||\breve{F}||^2_{H^m_{tan}(Q)} + ||\breve{G}||^2_{H^m(\Sigma)}\Big) \le C\Big(\tfrac{1}{\gamma}||F_\gamma||^2_{H^m_{tan}(Q_T)} + ||G_\gamma||^2_{H^m(\Sigma_T)}\Big)\end{aligned}$$

which concludes the proof of Theorem 15.

For the *homogeneous* IBVP with also *homogeneous boundary conditions*

$$\begin{aligned}&L_\gamma u_\gamma = F_\gamma\,, \quad \text{in } Q_T\,,\\ &Mu_\gamma = 0\,, \quad \text{on } \Sigma_T\,,\\ &u_{\gamma|\,t=0} = 0\,, \quad \text{in } \Omega\,,\end{aligned} \tag{91}$$

we have the following weaker version of Theorem 15.

Theorem 22. *Assume that A_i, B, for $i = 1, \dots, n$, are in $C^\infty(\overline{Q}_\infty)$, and that problem (91) satisfies assumptions (A)–(D) and (F); then for all $T > 0$ and $m \in \mathbb{N}$ there exist constants $C'_m > 0$ and γ'_m, with $\gamma'_m \le \gamma'_{m+1}$, such that for all $\gamma \ge \gamma'_m$, and for all $F_\gamma \in H^m_{tan}(Q_T)$ satisfying $\partial^h_t F_{\gamma|\,t=0} = 0$, for $h = 0, \dots, m-1$, the unique solution u_γ to (91) belongs to $H^m_{tan}(Q_T)$, and the a priori estimate*

$$\gamma||u_\gamma||_{H^m_{tan}(Q_T)} \le C'_m||F_\gamma||_{H^m_{tan}(Q_T)} \tag{92}$$

is fulfilled.

Proof. The proof follows the same lines of that of Theorem 15, disregarding all the estimates of boundary terms. □

Given the tangential regularity of the solution u of the homogeneous IBVP, as in Theorem 15 or Theorem 22, we could also prove its H^m_*-regularity. This can be done inductively by the following proposition that we give below without proof. An analogous proposition holds for the homogeneous problem (91) under assumptions (A)–(D) and (F).

Proposition 23. *Assume that problem (43) satisfies assumptions (A)–(E). For every integer $m \ge 2$, assume that $F_\gamma \in H^m_*(Q_T)$ and $G_\gamma \in H^m(\Sigma_T)$ satisfy (44). Let u_γ be the solution of (43) with data F_γ, G_γ and γ sufficiently large. Given any integer $2 \le q \le m$, assume that*

$$u_\gamma \in \mathcal{C}_T(H^q_{tan}) \cap \mathcal{C}_T(H^{q-1}_*) \quad \textit{with} \quad \partial_\nu(Pu_\gamma) \in \mathcal{C}_T(H^{q-2}_*)\,.$$

Then

$$u_\gamma \in \mathcal{C}_T(H^q_*) \quad \textit{with} \quad \partial_\nu(Pu_\gamma) \in \mathcal{C}_T(H^{q-1}_*)\,.$$

We will not follow this way because it doesn't seem to be useful in view of the nonhomogeneous IBVP. In fact, when dealing with the nonhomogeneous IBVP, as in [24] our strategy requires to subtract from u a regularized function which satisfies at least one more compatibility condition. Unfortunately, we can do it only for $m = 1$, starting with functions in H^1_*. Since $H^1_* = H^1_{tan}$, we can already cover this case with the regularity given in Theorem 15, without any need to increase the regularity to H^m_* with $m \ge 2$.

5. The nonhomogeneous IBVP. Proof for $m = 1$

For nonhomogeneous IBVP we mean problem (1)-(3) with initial data f different from zero. We distinguish between the case with nonhomogeneous boundary conditions, $G \neq 0$, and the other case with homogeneous boundary conditions, $G = 0$.

5.1. **Nonhomogeneous boundary conditions.** The main aim of this section is to prove the following theorem.

Theorem 24. *Assume* $(S_0, A_i, B) \in \mathcal{C}_T(H^\sigma_*) \times \mathcal{C}_T(H^\sigma_*) \times \mathcal{C}_T(H^{\sigma-2}_*)$, *for* $i = 1, \dots, n$, *where* $\sigma \geq [(n+1)/2] + 4$, *and that problem* (1)-(3) *obeys the assumptions* (A)–(E), (G). *Then for all* $F \in H^1_*(Q_T)$, $G \in H^1(\Sigma_T)$, $f \in H^1_*(\Omega)$, *with* $f^{(1)} \in L^2(\Omega)$, *satisfying the compatibility condition* $M_{|t=0} f_{|\partial\Omega} = G_{|t=0}$, *the unique solution* u *to* (1)–(3), *with data* (F, G, f), *belongs to* $\mathcal{C}_T(H^1_*)$ *and* $Pu_{|\,\Sigma_T} \in H^1(\Sigma_T)$.
Moreover, there exists a constant $C_1 > 0$ *such that* u *satisfies the a priori estimate*

$$||u||_{\mathcal{C}_T(H^1_*)} + ||Pu_{|\,\Sigma_T}||_{H^1(\Sigma_T)} \leq C_1 \left(|||f|||_{1,*} + ||F||_{H^1_*(Q_T)} + ||G||_{H^1(\Sigma_T)} \right). \tag{93}$$

Notice that Theorem 24 yields Theorem 2 for $m = 1$.

As a first step of the proof, we approximate the data with regularized functions satisfying one more compatibility condition.

Lemma 25. *Assume that problem* (1)-(3) *obeys the assumptions* (A)–(D). *Let* $F \in H^1_*(Q_T)$, $G \in H^1(\Sigma_T)$, $f \in H^1_*(\Omega)$, *with* $f^{(1)} \in L^2(\Omega)$, *such that* $M_{|t=0} f_{|\partial\Omega} = G_{|t=0}$.

Then there exist $F_k \in H^3(Q_T)$, $G_k \in H^3(\Sigma_T)$, $f_k \in H^3(\Omega)$, *such that* $M_{|t=0} f_k = G_{k|t=0}$, $\partial_t M_{|t=0} f_k + M_{|t=0} f_k^{(1)} = \partial_t G_{k|t=0}$ *on* $\partial\Omega$, *and such that* $F_k \to F$ *in* $H^1_*(Q_T)$, $G_k \to G$ *in* $H^1(\Sigma_T)$, $f_k \to f$ *in* $H^1_*(\Omega)$, $f_k^{(1)} \to f^{(1)}$ *in* $L^2(\Omega)$, *as* $k \to +\infty$.

Proof. For noncharacteristic homogeneous ($G = 0$) boundary conditions and the statement in standard Sobolev spaces H^m, a similar proposition has been proved in [24, Lemma 3.3]. Then it has been adapted to characteristic boundaries in [2, 20], again in standard Sobolev spaces H^m. In H^m_* spaces it seems that this can be done only for $m = 1$, see [29, Lemma 5.1]. The present adaptation to the nonhomogeneous case ($G \neq 0$) follows the same lines of the proof of [29, Lemma 5.1], so we will omit the details. □

Proof of Theorem 24. First we assume that the matrices S_0, A_j, B are of C^∞-class. Given the functions F_k, G_k, f_k as in Lemma 25, we first calculate through equation $Lu = F_k, u_{|t=0} = f_k$, the initial time derivatives $f_k^{(1)} \in H^2(\Omega)$, $f_k^{(2)} \in H^1(\Omega)$. Then we take a function $w_k \in H^3(Q_T)$ such that

$$w_{k|t=0} = f_k, \quad \partial_t w_{k|t=0} = f_k^{(1)}, \quad \partial^2_{tt} w_{k|t=0} = f_k^{(2)}.$$

Notice that this yields

$$(Lw_k)_{|t=0} = F_{k|t=0}, \quad \partial_t (Lw_k)_{|t=0} = \partial_t F_{k|t=0}. \tag{94}$$

Now we look for an approximated solution u_k of (1)-(3) with data F_k, G_k, f_k, of the form $u_k = v_k + w_k$, where v_k is solution to

$$\begin{array}{ll} Lv_k = F_k - Lw_k, & \text{in } Q_T \\ Mv_k = G_k - Mw_k, & \text{on } \Sigma_T \\ v_{k|t=0} = 0, & \text{in } \Omega. \end{array} \tag{95}$$

Let us denote again $L_\gamma = \gamma + L$, $u_{k\gamma} = e^{-\gamma t} u_k$, $v_{k\gamma} = e^{-\gamma t} v_k$ and so on. Then (95) is equivalent to

$$\begin{array}{ll} L_\gamma v_{k\gamma} = F_{k\gamma} - L_\gamma w_{k\gamma}, & \text{in } Q_T \\ Mv_{k\gamma} = G_{k\gamma} - Mw_{k\gamma}, & \text{on } \Sigma_T \\ v_{k\gamma|t=0} = 0, & \text{in } \Omega. \end{array} \tag{96}$$

We easily verify that (94) yields

$$(F_{k\gamma} - L_\gamma w_{k\gamma})_{|t=0} = 0, \quad \partial_t (F_{k\gamma} - L_\gamma w_{k\gamma})_{|t=0} = 0,$$

and $M_{|t=0} f_{k|\partial\Omega} = G_{k|t=0}$, $\partial_t M_{|t=0} f_{k|\partial\Omega} + M_{|t=0} f^{(1)}_{k|\partial\Omega} = \partial_t G_{k|t=0}$ yield

$$(G_{k\gamma} - Mw_{k\gamma})_{|t=0} = 0, \quad \partial_t (G_{k\gamma} - Mw_{k\gamma})_{|t=0} = 0.$$

Thus the sufficient conditions (44) hold for $h = 0, 1$, we may apply Theorem 15 for γ large enough, and find $v_k \in H^2_{tan}(Q_T)$, with $Pv_{k|\Sigma_T} \in H^2(\Sigma_T)$. Accordingly we infer $u_k \in H^2_{tan}(Q_T) \hookrightarrow \mathcal{C}_T(H^1_*)$, with

$Pu_{k|\Sigma_T} \in H^2(\Sigma_T)$. Moreover $u_k \in L^2(Q_T)$ solves

$$\begin{array}{ll} Lu_k = F_k, & \text{in } Q_T \\ Mu_k = G_k, & \text{on } \Sigma_T \\ u_{k|t=0} = f_k, & \text{in } \Omega. \end{array} \tag{97}$$

Take a covering $\{U_j\}_{j=0}^l$ of Ω and a partition of unity $\{\psi_j\}_{j=0}^l$ subordinate to this covering, as in Section 2. In each patch $U_j, j = 1, \dots, l$, we find the unitary matrix $T_j(x,t)$ of Lemma 17; we also set $T_0 := I_n$. For each $j = 0, \dots, l$ we denote $u_k^j(x,t) := T_j(x,t)\psi_j(x)u_k(x,t)$ and define similarly F_k^j, G_k^j, f_k^j. In local coordinates each u_k^j solves a problem of the form

$$\begin{array}{ll} L^j u_k^j = F_k^j + K^j u_k, & \text{in } \mathbb{R}^n_+ \times]0,T[, \\ \widetilde{M} u_k^j = G_k^j, & \text{on } \{x_1 = 0\} \times \mathbb{R}^{n-1}_{x'} \times]0,T[, \\ u^j_{k|t=0} = f_k^j, & \text{in } \mathbb{R}^n_+, \end{array}$$

with $L^j, \widetilde{M}$ as in (54). The boundary matrix $-A_1^j$ has the block form as in (51), (52) and $\mathrm{K}^j u$ is a certain operator of order 0 (see [29]) that can be absorbed in B^j by considering the compound system of all $j = 0, \dots, l$. After doing that we are lead to consider the problem

$$\begin{array}{ll} L^j u_k^j = F_k^j, & \text{in } \mathbb{R}^n_+ \times]0,T[, \\ \widetilde{M} u_k^j = G_k^j, & \text{on } \{x_1 = 0\} \times \mathbb{R}^{n-1}_{x'} \times]0,T[, \\ u^j_{k|t=0} = f_k^j, & \text{in } \mathbb{R}^n_+. \end{array} \tag{98}$$

We look for the problem solved by $Zu_k = (Z_1 u_k, \dots, Z_{n+1}u_k) \in H^1_{tan}(Q_T) = H^1_*(Q_T)$ (where $Z_{n+1} = \partial_t$). As already observed by Rauch [22], there exist matrices $\Gamma_\beta, \Gamma_0, \Psi$ such that

$$[L^j, Z_i] = -\textstyle\sum_{|\beta|=1} \Gamma_\beta Z^\beta + \Gamma_0 + \Psi L^j, \qquad i = 1, \dots, n+1. \tag{99}$$

As we will see more in detail in Lemma 41, Γ_β looses at most one normal and one tangential derivative w.r.t. the A_j's (i.e. a weight 3 in H^m_* spaces) and Γ_0, Ψ loose at most one tangential derivative (weight 1 in H^m_* spaces). However, for smooth matrices A_j, B we do not need to care about that loss of regularity.

For higher order commutators one proves that for each α there exist matrices $\Gamma_{\alpha,\beta}, \Gamma_{\alpha,\gamma}, \Psi_{\alpha,\gamma}$ such that

$$[L^j, Z^\alpha] = -\textstyle\sum_{|\beta|=|\alpha|} \Gamma_{\alpha,\beta} Z^\beta + \sum_{|\gamma|<|\alpha|} \Gamma_{\alpha,\gamma} Z^\gamma + \sum_{|\gamma|<|\alpha|} \Psi_{\alpha,\gamma} Z^\gamma L^j. \tag{100}$$

These commutators will be detailed in the proof of Subsection 6.1.

Again, for the sake of simplicity, we remove the indices j and write M instead of $\widetilde{M}$. Applying the operators Z_i to (98) and taking account of (99), we infer that Zu_k solves problem

$$\begin{array}{ll} LZ_i u_k + \sum_{|\beta|=1} \Gamma_\beta Z^\beta u_k = (Z_i + \Psi)F_k + \Gamma_0 u_k, & \text{in } \mathbb{R}^n_+ \times]0,T[, \\ MZ_i u_k = Z_i G_k, & \text{on } \{x_1 = 0\} \times \mathbb{R}^{n-1}_{x'} \times]0,T[, \\ Z_i u_{k|t=0} = Z_i f_k, & \text{in } \mathbb{R}^n_+. \end{array} \tag{101}$$

Applying the a priori estimate (5) to a difference of solutions $u_k - u_h$ of problems (97), (101) readily gives

$$\begin{aligned} &||u_k - u_h||_{\mathcal{C}_T(H^1_*)} + ||P(u_k - u_h)_{|\Sigma_T}||_{H^1(\Sigma_T)} \\ &\le C \left(|||f_k - f_h|||_{1,*} + ||F_k - F_h||_{H^1_*(Q_T)} + ||G_k - G_h||_{H^1(\Sigma_T)} \right). \end{aligned}$$

From Lemma 25 we infer that $\{u_k\}$ is a Cauchy sequence in $\mathcal{C}_T(H^1_*)$, and $\{Pu_{k|\,\Sigma_T}\}$ is a Cauchy sequence in $H^1(\Sigma_T)$. Therefore there exists a function in $\mathcal{C}_T(H^1_*)$ which is the limit of $\{u_k\}$. Passing to the limit in (97) as $k \to \infty$, we see that this function is a solution to (1)-(3). The uniqueness of the L^2 solution yields $u \in \mathcal{C}_T(H^1_*)$ and $Pu_{|\,\Sigma_T} \in H^1(\Sigma_T)$. Applying the a priori estimate (5) to the solutions u_k of problems (97), (101), and passing to the limit finally gives (93).

Up to now we have considered matrices S_0, A_j, B of C^∞-class. Now we wish to solve the problem with coefficients with finite regularity as in Theorem 2, by a density argument.

Given matrices S_0, A_j, B with the properties prescribed in the statement of Theorem 24, let us take approximating sequences $S_0^{(k)}, A_j^{(k)}, B^{(k)}$ in C^∞, such that $S_0^{(k)} \to S_0$, $A_j^{(k)} \to A_j$ in $\mathcal{C}_T(H^\sigma_*)$, and

$B^{(k)} \to B$ in $\mathcal{C}_T(H^{\sigma-2}_*)$, as $k \to \infty$, where $\sigma \geq [(n+1)/2]+4$. We may also assume that $S_0^{(k)}$ is definite positive, and that the new boundary matrix in local coordinates has the same properties as in (51), (52). This yields operators $L^{(k)}$ converging to L, where assumptions (A)–(D) are still satisfied.

In this context now we prove the following result. To avoid overloading with the introduction of a new notation, we use the same symbols of Lemma 25.

Lemma 26. *There exist $F_k \in H^3(Q_T)$, $G_k \in H^3(\Sigma_T)$, $f_k \in H^3(\Omega)$, such that $M_{|t=0} f_k = G_{k|t=0}$ on $\partial\Omega$, and such that $F_k \to F$ in $H^1_*(Q_T)$, $G_k \to G$ in $H^1(\Sigma_T)$, $f_k \to f$ in $H^1_*(\Omega)$, $f_k^{(1)} \to f^{(1)}$ in $L^2(\Omega)$, as $k \to +\infty$, where now $f_k^{(1)}$ is defined by*

$$f_k^{(1)} + \sum_{i=1}^n A^{(k)}_{i|t=0} \partial_i f_k + B^{(k)}_{|t=0} f_k = F_{k|t=0} \qquad \text{in } \Omega.$$

Proof. The proof is quite similar to that of Lemma 25, somehow easier because here we require only one compatibility condition. For all details we refer again to [29, Lemma 5.1]. □

We consider the problems

$$\begin{array}{ll} L^{(k)} u^{(k)} = F_k, & \text{in } Q_T \\ M u^{(k)} = G_k, & \text{on } \Sigma_T \\ u^{(k)}_{|t=0} = f_k, & \text{in } \Omega. \end{array}$$

The operator $L^{(k)}$ has C^∞ coefficients and the data have the required regularity and enjoy the compatibility condition of order 0. Therefore we may apply the previous step of the proof and find solutions $u^{(k)} \in \mathcal{C}_T(H^1_*)$. This is the point where we use assumption (G).

Going back again to the estimate (93), we observe that the constant C_1 depends on the norms of the S_0, A_j's in $Lip(\overline{Q}_T)$, and the norm of B in $L^\infty(Q_T)$ at the L^2-level, and also on the L^∞ norms of $\Gamma_\beta, \Gamma_0, \Psi$ at the H^1_*-level.

Since $S_0^{(k)}, A_j^{(k)}$ are uniformly bounded in $\mathcal{C}_T(H_*^{[(n+1)/2]+3})$, applying the imbedding Theorem 34 shows that they are uniformly bounded in $C^0([0,T];W^{1,\infty}) \cap C^1([0,T];L^\infty)$, and therefore in $Lip(\overline{Q}_T)$. Similarly one infers that the $B^{(k)}$'s are uniformly bounded in $L^\infty(Q_T)$.

Again, since $S_0^{(k)}, A_j^{(k)}$ are uniformly bounded in $\mathcal{C}_T(H_*^{[(n+1)/2]+4})$, by Lemma 41 the approximating matrices $\Gamma_\beta^{(k)}$ are uniformly bounded in $C([0,T];H_*^{[(n+1)/2]+1})$. We may apply the imbedding Theorem 34 and obtain the uniform boundedness in $L^\infty(Q_T)$. Similarly we infer the uniform boundedness in $L^\infty(Q_T)$ of $\Gamma_0^{(k)}, \Psi^{(k)}$.

Then the $u^{(k)}$'s satisfy the apriori estimate (93) with uniformly bounded constants $C_1^{(k)}$. Therefore the sequence $\{u^{(k)}\}$ is bounded in $\mathcal{C}_T(H^1_*)$ with $\{Pu^{(k)}_{|\Sigma_T}\}$ bounded in $H^1(\Sigma_T)$. Passing to a subsequence we get a solution $u \in \mathcal{L}^\infty_T(H^1_*)$ with $Pu_{|\Sigma_T} \in H^1(\Sigma_T)$. The uniqueness of the solution yields the convergence of the whole sequence. The strong continuity in time follows by adapting Majda's approach [15].

This completes the proof of Theorem 24.

5.2. **Homogeneous boundary conditions.** In case of homogeneous boundary conditions, when assumption (E) is substituted by (F), we may obtain a result similar to Theorem 24, but with no information about the trace of the solution at the boundary, see Remark 4. The proof is similar to that of Theorem 24 and so we will omit it.

Theorem 27. *Assume $(S_0, A_i, B) \in \mathcal{C}_T(H^\sigma_*) \times \mathcal{C}_T(H^\sigma_*) \times \mathcal{C}_T(H^{\sigma-2}_*)$, for $i = 1, \dots, n$, where $\sigma \geq [(n+1)/2]+4$, and that problem (1)-(3) obeys the assumptions (A)–(D), (F), (G). Then for all $F \in H^1_*(Q_T)$, $f \in H^1_*(\Omega)$, with $f^{(1)} \in L^2(\Omega)$, satisfying the compatibility condition $M_{|t=0} f_{|\partial\Omega} = 0$, the unique solution u to (1)–(3), with data $(F, G = 0, f)$, belongs to $\mathcal{C}_T(H^1_*)$. Moreover, there exists a constant $C_1' > 0$ such that u satisfies the a priori estimate*

$$||u||_{\mathcal{C}_T(H^1_*)} \leq C_1' \left(|||f|||_{1,*} + ||F||_{H^1_*(Q_T)} \right). \tag{102}$$

6. The nonhomogeneous IBVP. Proof for $m \geq 2$

The proof proceeds by induction. Assume that Theorem 2 holds up to $m-1$. Let $f \in H^m_*(\Omega)$, $F \in H^m_*(Q_T)$, $G \in H^m(\Sigma_T)$, with $f^{(k)} \in H^{m-k}_*(\Omega)$, $k = 1, \cdots, m$, and assume also that the compatibility conditions (8) hold up to order $m-1$. By the inductive hypothesis there exists a unique solution u of problem (1)-(3) such that $u \in \mathcal{C}_T(H^{m-1}_*)$.

In order to show that $u \in \mathcal{C}_T(H^m_*)$, we have to increase the regularity of u by order one, that is by one more tangential derivative and, if m is even, also by one more normal derivative. This can be done as in [27, 29], with the small change of the elimination of the auxiliary system (introduced in [27, 29]) as in [5, 31]. At every step we can estimate some derivatives of u through equations where in the right-hand side we can put other derivatives of u that have already been estimated at previous steps. The reason why the main idea in [27] works, even though here we do not have maximally nonnegative boundary conditions, is that for the increase of regularity we consider the system (106) of equations for purely tangential derivatives of the type of (1)-(3), where we can use the inductive assumption, and other systems (108), (109) of equations for mixed tangential and normal derivatives where the boundary matrix vanishes identically, so that no boundary condition is needed and we can apply an energy method, under the assumption of the symmetrizable system. Without entering in too many details we briefly describe the different steps of the proof, for the reader's convenience.

As before, we take a covering $\{U_j\}_{j=0}^l$ of Ω and a partition of unity $\{\psi_j\}_{j=0}^l$ subordinate to this covering. In each patch $U_j, j = 1, \ldots, l$, we find the unitary matrix $T_j(x,t)$ of Lemmata 16 and 17; we also set $T_0 := I_n$. For each $j = 0, \ldots, l$ we denote $u^j(x,t) := T_j(x,t)\psi_j(x)u(x,t)$ and define similarly F^j, G^j, f^j. In local coordinates each u^j solves a problem of the form

$$
\begin{array}{ll}
L^j u^j = F^j + K^j u, & \text{in } \mathbb{R}^n_+ \times]0,T[, \\
\widetilde{M} u^j = G^j, & \text{on } \{x_1 = 0\} \times \mathbb{R}^{n-1}_{x'} \times]0,T[, \\
u^j_{|t=0} = f^j, & \text{in } \mathbb{R}^n_+,
\end{array}
$$

with $L^j, \widetilde{M}$ as in (54). The boundary matrix $-A^j_1$ has the block form as in (51), (52) and $\mathrm{K}^j u$ is a certain operator of order 0 (see [29]) that can be absorbed in B^j by considering the compound system of all $j = 0, \ldots, l$. After doing that we are lead to consider the problem

$$
\begin{array}{ll}
L^j u^j = F^j, & \text{in } \mathbb{R}^n_+ \times]0,T[, \\
\widetilde{M} u^j = G^j, & \text{on } \{x_1 = 0\} \times \mathbb{R}^{n-1}_{x'} \times]0,T[, \\
u^j_{|t=0} = f^j, & \text{in } \mathbb{R}^n_+.
\end{array}
\tag{103}
$$

Again, for the sake of simplicity, we will remove the indices j and write M instead of $\widetilde{M}$. Hereafter we will denote $Z = (Z_1, \ldots, Z_{n+1}), Z_{n+1} = \partial_t$.

6.1. **Purely tangential regularity.** Let us start by considering all the tangential derivatives $Z^\alpha u$, $|\alpha| = m-1$. We decompose $\partial_1 u = \begin{pmatrix} \partial_1 u^I \\ \partial_1 u^{II} \end{pmatrix}$. By inverting $A^{I,I}_1$ in $(103)_1$, we can write $\partial_1 u^I$ as the sum of tangential derivatives by

$$
\partial_1 u^I = \Lambda Z u + R \tag{104}
$$

where

$$
\begin{aligned}
\Lambda Z u &= (A^{I,I}_1)^{-1} \big[(A_{n+1} Z_{n+1} u + \sum_{j=2}^n A_j Z_j u)^I + A^{I,II}_1 \partial_1 u^{II} \big], \\
R &= (A^{I,I}_1)^{-1} (Bu - F)^I.
\end{aligned}
$$

Here and below, everywhere it is needed, we use the fact that, if a matrix A vanishes on $\{x_1 = 0\}$, we can write $A\partial_1 u = HZ_1 u$, where H is a suitable matrix such that $||H||_{H^{s-2}_*(\Omega)} \leq c||A||_{H^s_*(\Omega)}$ see Lemmata 39 and 40 in the Appendix B; this trick transforms some normal derivatives into tangential derivatives. We obtain $\Lambda \in \mathcal{C}_T(H^{s-2}_*)$.

Applying the operator Z^α to (103), with $\alpha=(\alpha',\alpha_{n+1})$, $\alpha'=(\alpha_1,\cdots,\alpha_n)$, and substituting (104) gives equation (5.3) in [27], that is

$$\begin{aligned}L(Z^\alpha u)&+\sum_{|\gamma|=|\alpha|-1}(ZA_{n+1}Z_{n+1}+\sum_{j=2}^n ZA_jZ_j)Z^\gamma u+\sum_{|\gamma|=|\alpha|-1}ZA_1\begin{pmatrix}\Lambda Z(Z^\gamma u)\\0\end{pmatrix}\\&-\alpha_1A_1\begin{pmatrix}\Lambda Z(Z_1^{\alpha_1-1}Z_2^{\alpha_2}\cdots Z_{n+1}^{\alpha_{n+1}}u)\\0\end{pmatrix}\\&+\Big(\sum_{|\gamma|=|\alpha|-1}ZA_1Z^\gamma-\alpha_1A_1Z_1^{\alpha_1-1}Z_2^{\alpha_2}\cdots Z_{n+1}^{\alpha_{n+1}}\Big)\begin{pmatrix}0\\\partial_1u^{II}\end{pmatrix}=F_\alpha,\end{aligned}\tag{105}$$

with $F_\alpha\in H^1_*(Q_T)$, see [27] for its explicit expression. Equation (105) takes the form $(L+\overline{B})Z^\alpha u=F_\alpha$ with $\overline{B}\in\mathcal{C}_T(H^{s-3}_*)$. Notice that $s-3\geq[(n+1)/2]+2$ as required in Theorem 24.

Then we consider the problem satisfied by the vector of all tangential derivatives $Z^\alpha u$ of order $|\alpha|=m-1$. From (105) this problem takes the form

$$\begin{array}{ll}(\mathcal{L}+\mathcal{B})Z^\alpha u=\mathcal{F} & \text{in } \mathbb{R}^n_+\times]0,T[,\\ \mathcal{M}Z^\alpha u=Z^\alpha G & \text{on } \{x_1=0\}\times\mathbb{R}^{n-1}_{x'}\times]0,T[,\\ Z^\alpha u|_{t=0}=\tilde f & \text{in } \mathbb{R}^n_+,\end{array}\tag{106}$$

where

$$\mathcal{L}=\begin{pmatrix}L&&\\&\ddots&\\&&L\end{pmatrix},\qquad \mathcal{M}=\begin{pmatrix}M&&\\&\ddots&\\&&M\end{pmatrix},$$

$\mathcal{B}\in\mathcal{C}_T(H^{s-3}_*)$ is a suitable linear operator and $\mathcal{F}$ is the vector of all right-hand sides F_α. The initial datum $\tilde f$ is the vector of functions $Z^{\alpha'}f^{(\alpha_{n+1})}$.

We have $\mathcal{F}\in H^1_*(Q_T)$, $\tilde f\in H^1_*$, $Z^\alpha G\in H^1(\Sigma_T)$. Moreover the data satisfy the compatibility conditions of order 0. By applying Theorem 24, we infer $Z^\alpha u\in\mathcal{C}_T(H^1_*)$, for all $|\alpha|=m-1$.

6.2. **Tangential and one normal derivatives.** We apply to the part II of $(103)_1$ the operator $Z^\beta\partial_1$, with $|\beta|=m-2$. We obtain equation (28) in [5], that is

$$\begin{aligned}&\big[(L+\partial_1A_1)Z^\beta+\sum_{|\gamma|=|\beta|-1}(ZA_0\partial_t+\sum_{j=1}^n ZA_j\partial_j)Z^\gamma\\&-\beta_1A_1\partial_1Z_1^{\beta_1-1}Z_2^{\beta_2}\cdots Z_{n+1}^{\beta_{n+1}}\big]^{II,II}\partial_1u^{II}=\mathcal{G},\end{aligned}\tag{107}$$

where the exact expression of $\mathcal{G}$ may be found in [5]. It is important to observe that $\mathcal{G}$ contains only tangential derivatives of order at most m. Hence, we can estimate it by using the previous step and infer $\mathcal{G}\in L^2(Q_T)$. Using (104) again, we write (107) as

$$(\tilde{\mathcal{L}}+\tilde{\mathcal{C}})Z^\beta\partial_1u^{II}=\mathcal{G},\tag{108}$$

where

$$\tilde{\mathcal{L}}=\begin{pmatrix}\tilde L&&\\&\ddots&\\&&\tilde L\end{pmatrix}$$

with $\tilde L=A_0^{II,II}\partial_t+\sum_{j=1}^n A_j^{II,II}\partial_j$ and where $\tilde{\mathcal{C}}\in\mathcal{C}_T(H^{s-2}_*)$ is a suitable linear operator. Here a crucial point is that (108) is a transport-type equation, because the boundary matrix of $\tilde{\mathcal{L}}$ vanishes at $\{x_1=0\}$. Thus we do not need any boundary condition. We infer that equation (108) has a unique solution $Z^\beta\partial_1u^{II}\in C_T(L^2):=C([0,T];L^2(\mathbb{R}^n_+))$, for all $|\beta|=m-2$. Using (104) again, we deduce $Z^\beta\partial_1u\in C_T(L^2)$, for all $|\beta|=m-2$.

6.3. **Normal derivatives.** The last step is again by induction, as in [27], page 867, (ii). For convenience of the reader, we provide a brief sketch of the proof.

Suppose that for some fixed k, with $1 \le k < [m/2]$, it has already been shown that $Z^\alpha \partial_1^h u$ belongs to $C_T(L^2)$, for any h and α such that $h = 1, \cdots, k$, $|\alpha| + 2h \le m$. From (104) it immediately follows that $Z^\alpha \partial_1^{k+1} u^I \in C_T(L^2)$. It rests to prove that $Z^\alpha \partial_1^{k+1} u^{II} \in C_T(L^2)$.

We apply operator $Z^\alpha \partial_1^{k+1}$, $|\alpha| + 2k = m - 2$, to the part II of $(103)_1$ and obtain an equation similar to (108) of the form

$$(\tilde{\mathcal{L}} + \tilde{\mathcal{C}}_k) Z^\alpha \partial_1^{k+1} u^{II} = \mathcal{G}_k, \tag{109}$$

where $\tilde{\mathcal{C}}_k \in \mathcal{C}_T(H_*^{s-3})$ is a suitable linear operator. The right-hand side $\mathcal{G}_k$ contains derivatives of u of order m (in H_*^m, i.e. counting 1 for each tangential derivative and 2 for normal derivatives), but contains only normal derivatives that have already been estimated. We infer $\mathcal{G}_k \in L^2(Q_T)$. Again it is crucial that the boundary matrix of $\tilde{\mathcal{L}}$ vanishes at $\{x_1 = 0\}$. We infer that the solution $Z^\alpha \partial_1^{k+1} u^{II}$ is in $C_T(L^2)$ for all α, k with $|\alpha| + 2k = m - 2$. By repeating this procedure we obtain the result for any $k \le [m/2]$, hence $u \in \mathcal{C}_T(H_*^m)$.

Remark 28. *In order to show that $u \in \mathcal{C}_T(H_*^m)$, an analysis of the definition of spaces H_*^m shows that it is sufficient to prove $Zu \in \mathcal{C}_T(H_*^{m-1})$, and that $\partial_1^{m/2} u \in \mathcal{C}_T(L^2)$ when m is even. In this respect, a simple proof of $Zu \in \mathcal{C}_T(H_*^{m-1})$ could seem as follows. Let us apply the operators Z_i to* (103) *and use* (99)*; this gives the problem*

$$\begin{array}{ll} LZ_i u + \sum_{|\beta|=1} \Gamma_\beta Z^\beta u = (Z_i + \Psi) F + \Gamma_0 u, & \textit{in } \mathbb{R}^n_+ \times]0, T[, \\ M Z_i u = Z_i G, & \textit{on } \{x_1 = 0\} \times \mathbb{R}^{n-1}_{x'} \times]0, T[, \\ Z_i u_{|t=0} = Z_i f, & \textit{in } \mathbb{R}^n_+, \end{array} \tag{110}$$

for $i = 1, \ldots, n+1$. Lemma 41 shows that $\Gamma_\beta \in \mathcal{C}_T(H_*^{s-3})$, $\Gamma_0 \in \mathcal{C}_T(H_*^{s-1})$, $\Psi \in \mathcal{C}_T(H_*^{s-1})$. Using the result of Theorem 38, we easily verify that $(Z_i + \Psi) F + \Gamma_0 u \in H_*^{m-1}(Q_T)$. Moreover it is clear that $Z_i G \in H^{m-1}(\Sigma_T)$, $Z_i f \in H_*^{m-1}(\Omega)$ with $Z_i f^{(k)} \in H_*^{m-k-1}(\Omega)$, $k = 1, \cdots, m-1$, and the compatibility conditions for (110) *hold up to order $m - 2$.*

Unfortunately, it is also clear that we can't apply Theorem 2 (with order $m - 1$), because Γ_β is not smooth enough for being absorbed in the 0-th order term of the operator, required to be in $\mathcal{C}_T(H_*^{s-1})$. In fact, with this approach we are not using well the compensation in the loss of derivatives that appears when differentiating products of functions.

The apriori estimate (9) follows from (93) (namely estimate (9) in case $m = 1$, proved in Theorem 24) applied to the solution of (106), plus standard L^2 energy estimates for equations (108) and (109), and the direct estimate of the normal derivative of u by tangential derivatives via (104). All products of functions are estimated in spaces H_*^m by the rules given in Theorem 38 and Lemmata 39 and 40 in the Appendix. We refer the reader to [5, 27, 29] for all details.

We observe that the constant C_m in (9) not only depends on the norms of the S_0, A_j's in $\mathcal{C}_T(H_*^s)$, and the norm of B in $\mathcal{C}_T(H_*^{s-1})$ if $m \le s - 1$, or in $\mathcal{C}_T(H_*^s)$ if $m = s$. It depends also on the constant of positive definiteness of the symmetrizing matrix S_0, and on an upper bound for the inverse of the nonsingular part of the boundary matrix (e.g. a constant C such that $|(\widehat{A}_1^{I,I})^{-1}| \le C$ in (51), on each patch).

This concludes the proof of Theorem 2.

6.4. **Homogeneous boundary conditions.** The proof of Theorem 3 for homogeneous boundary conditions, when assumption (E) is substituted by (F), is similar to that of Theorem 2, and so we will omit it.

Appendix A. Proof of Proposition 12

Let $\widetilde{\chi}_\varepsilon$ be the function defined in (40). We begin giving some properties of its Fourier transform. Firstly, we notice that from formula (38), the following estimates for the derivatives of $\widehat{\chi}$ can be easily derived

$$\partial^\alpha \widehat{\chi}(\xi) = O(|\xi|^{p-|\alpha|}), \quad \xi \to 0, \tag{111}$$

for all $\alpha \in \mathbb{N}^n$ with $|\alpha| \le p$.
In the sequel, we denote by ξ' the Fourier dual variables of $y' \in \mathbb{R}^n$.

Lemma 29. *Let $\widetilde{\chi}_\varepsilon$ be defined in (40). There exists a positive constant C such that*

$$|\widehat{\widetilde{\chi}}_\varepsilon(\xi')| \le C\varepsilon^p(\sum_{j<p}|\xi'|^{p-j}+1)\,, \quad \forall \xi' \in \mathbb{R}^n\,, \forall \varepsilon \in]0,1[\,. \tag{112}$$

Proof. By definition, we find for $\widehat{\widetilde{\chi}}_\varepsilon$ the following formula

$$\widehat{\widetilde{\chi}}_\varepsilon(\xi') = \int_{\mathbb{R}^{n+1}} e^{-i(y_1,y')\cdot(0,\varepsilon\xi')}e^{-\frac{\varepsilon y_1}{2}}\chi(y)dy. \tag{113}$$

A Taylor's expansion of $e^{-\frac{\varepsilon y_1}{2}}$ gives

$$e^{-\frac{\varepsilon y_1}{2}} = \sum_{j<p}\frac{(-\varepsilon y_1/2)^j}{j!} + \frac{1}{p!}(-\varepsilon y_1/2)^p e^{-\frac{\theta\varepsilon y_1}{2}}$$

for a suitable $\theta \in]0,1[$. Inserting in (113) we get

$$\widehat{\widetilde{\chi}}_\varepsilon(\xi') = \sum_{j<p} C_j\varepsilon^j\widehat{y_1^j\chi}(0,\varepsilon\xi') + C_p\varepsilon^p\int_{\mathbb{R}^{n+1}} e^{-i(y_1,y')\cdot(0,\varepsilon\xi')}y_1^p e^{-\frac{\theta\varepsilon y_1}{2}}\chi(y)dy,$$

with positive constants C_j, $j \le p$, independent of ε. From $\widehat{y_1^j\chi}(\xi) = (i\partial_{\xi_1})^j\widehat{\chi}(\xi)$ and using estimates (111) (and that $\widehat{\chi}$ is rapidly decreasing), we derive

$$\begin{aligned}|\widehat{\widetilde{\chi}}_\varepsilon(\xi')| &\le \sum_{j<p} C_j\varepsilon^j|\partial^j_{\xi_1}\widehat{\chi}(0,\varepsilon\xi')| + C_p\varepsilon^p\int_{\operatorname{supp}\chi}|y_1|^p e^{\frac{|y_1|}{2}}|\chi(y)|dy\\ &\le \sum_{j<p} C_j\varepsilon^j|\varepsilon\xi'|^{p-j} + C_p\varepsilon^p M \le C\varepsilon^p(\sum_{j<p}|\xi'|^{p-j}+1)\,, \quad \forall\, \xi' \in \mathbb{R}^n\,,\end{aligned}$$

with $C > 0$ independent of ε. □

Lemma 30. *For all $N > 0$ there exists a constant $C_N > 0$ such that*

$$(1+|\varepsilon\xi'|^2)^N|\widehat{\widetilde{\chi}}_\varepsilon(\xi')| \le C_N\,, \quad \forall \xi' \in \mathbb{R}^n\,,\ \forall \varepsilon \in]0,1[\,. \tag{114}$$

Proof. By Newton's formula, we compute

$$(1+|\varepsilon\xi'|^2)^N = \sum_{|\alpha'|\le N} C_{\alpha'}(\varepsilon\xi')^{2\alpha'},$$

for suitable constants $C_{\alpha'} > 0$. To derive (114), it is enough to bound $(\varepsilon\xi')^{2\alpha'}\widehat{\widetilde{\chi}}_\varepsilon(\xi')$ uniformly in ε. Using formula (40) and integrating by parts, we obtain

$$(\varepsilon\xi')^{2\alpha'}\widehat{\widetilde{\chi}}_\varepsilon(\xi') = (-1)^{|\alpha'|}\int_{\mathbb{R}^{n+1}} e^{-i\varepsilon\xi'\cdot y'}\partial^{2\alpha'}_{y'}\chi(y_1,y')e^{-\frac{\varepsilon y_1}{2}}dy_1dy',$$

hence

$$|(\varepsilon\xi')^{2\alpha'}\widehat{\widetilde{\chi}}_\varepsilon(\xi')| \le \int_{\operatorname{supp}\chi}|\partial^{2\alpha'}_{y'}\chi(y_1,y')|e^{\frac{|y_1|}{2}}dy_1dy' =: C_{\alpha'}.$$

Inequality (114) immediately follows summing over all $\alpha' \in \mathbb{N}^n$ with $|\alpha'| \le N$. □

We note that the interesting feature of both estimates (112), (114) consists of the uniformity with respect to ε: this seems to be not a trivial consequence of the rapid decrease of $\widehat{\widetilde{\chi}}_\varepsilon$, due to the kind of dependence on ε displayed by (40).

Proof of Proposition 12. The proof follows by adapting that of Theorem 2.4.1 in Hörmander [13]. We divide it in two steps.
1st step: Right-hand estimate in (41).
Using Parseval's formula we obtain

$$\int_0^1 ||u*\widetilde{\chi}_\varepsilon||^2_{L^2(\mathbb{R}^n)}\varepsilon^{-2k}\left(1+\frac{\delta^2}{\varepsilon^2}\right)^{-1}\frac{d\varepsilon}{\varepsilon} = (2\pi)^{-n}\int_{\mathbb{R}^n}|\widehat{u}(\xi')|^2F(\xi',\delta)d\xi', \tag{115}$$

where

$$F(\xi',\delta)=\int_0^1|\widehat{\widetilde{\chi}}_\varepsilon(\xi')|^2\varepsilon^{-2k}\left(1+\frac{\delta^2}{\varepsilon^2}\right)^{-1}\frac{d\varepsilon}{\varepsilon}.$$

Using Lemmata 29, 30 and arguing in a similar way as in [13, Theorem 2.4.1], we deduce the following estimate

$$F(\xi',\delta)\le C(1+|\xi'|^2)^k(1+|\delta\xi'|^2)^{-1}\,,\quad \forall\delta\in]0,1]\,,\ \forall\,\xi'\in\mathbb{R}^n\,. \tag{116}$$

Then, (115) and (116) immediately give

$$\begin{aligned}&\textstyle\int_0^1||u*\widetilde{\chi}_\varepsilon||^2_{L^2(\mathbb{R}^n)}\varepsilon^{-2k}\left(1+\frac{\delta^2}{\varepsilon^2}\right)^{-1}\frac{d\varepsilon}{\varepsilon}\\&\le C(2\pi)^{-n}\textstyle\int_{\mathbb{R}^n}|\widehat{u}(\xi')|^2(1+|\xi'|^2)^k(1+|\delta\xi'|^2)^{-1}d\xi'=C||u||^2_{\mathbb{R}^n,k-1,\delta}\end{aligned} \tag{117}$$

from which the right-hand inequality in (41) follows, observing that

$$||u||_{H^{k-1}(\mathbb{R}^n)}:=||u||_{\mathbb{R}^n,k-1,1}\le||u||_{\mathbb{R}^n,k-1,\delta}\quad\forall\,0<\delta\le 1.$$

2$^{\text{nd}}$ step: Left-hand estimate in (41).
We observe that our estimate can be obtained as a consequence of the following two facts:

h1. There exist $M\ge 1$ and $C_1>0$ such that

$$F(\xi',\delta)\ge C_1(1+|\xi'|^2)^k(1+|\delta\xi'|^2)^{-1}\,,\quad\forall\,\xi'\in\mathbb{R}^n\;:\;|\xi'|\ge M\,,\ \forall\delta\in]0,1];$$

h2. There exists $C_2>0$ such that

$$(1+|\xi'|^2)^{k-1}\ge C_2(1+|\xi'|^2)^k(1+|\delta\xi'|^2)^{-1}\,,\quad\forall\,\xi'\in\mathbb{R}^n\;:\;|\xi'|\le M\,,\ \forall\delta\in]0,1],$$

with the same M involved in **h1**.

Indeed, if **h1**, **h2** are fulfilled, we get

$$\begin{aligned}&\textstyle\int_0^1||u*\widetilde{\chi}_\varepsilon||^2_{L^2(\mathbb{R}^n)}\varepsilon^{-2k}\left(1+\frac{\delta^2}{\varepsilon^2}\right)^{-1}\frac{d\varepsilon}{\varepsilon}+||u||^2_{\mathbb{R}^n,k-1}\\&=(2\pi)^{-n}\textstyle\int_{\mathbb{R}^n}|\widehat{u}(\xi')|^2F(\xi',\delta)d\xi'+(2\pi)^{-n}\int_{\mathbb{R}^n}|\widehat{u}(\xi')|^2(1+|\xi'|^2)^{k-1}d\xi'\\&\ge(2\pi)^{-n}C_1\textstyle\int_{|\xi'|\ge M}|\widehat{u}(\xi')|^2(1+|\xi'|^2)^k(1+|\delta\xi'|^2)^{-1}d\xi'\\&+(2\pi)^{-n}C_2\textstyle\int_{|\xi'|\le M}|\widehat{u}(\xi')|^2(1+|\xi'|^2)^k(1+|\delta\xi'|^2)^{-1}d\xi'\\&\ge\min\{C_1,C_2\}||u||^2_{\mathbb{R}^n,k-1,\delta}.\end{aligned}$$

It remains to prove that **h1** and **h2** are satisfied. **h2** is trivial; hence we concentrate now to prove that **h1** is fulfilled.
Since $\chi\in C_0^\infty(\mathbb{R}^{n+1})$, Paley-Wiener's theorem gives that its Fourier transform $\widehat{\chi}(\xi)$, for $\xi\in\mathbb{R}^{n+1}$, admits an extension to the complex space $\mathbb{C}^{n+1}$, denoted again by $\widehat{\chi}$, which is an entire analytic function on $\mathbb{C}^{n+1}$. By the definition of functions $\widetilde{\chi}_\varepsilon$, it follows that

$$\widehat{\widetilde{\chi}}_\varepsilon(\xi')=\widehat{\chi}(-\frac{i\varepsilon}{2},\varepsilon\xi')\,,\quad\forall\,\xi'\in\mathbb{R}^n\,.$$

We use this identity to write $F(\xi',\delta)$, appearing in formula (115), as

$$F(\xi',\delta)=\int_0^1|\widehat{\chi}(-\frac{i\varepsilon}{2},\varepsilon\xi')|^2\varepsilon^{-2k}\left(1+\frac{\delta^2}{\varepsilon^2}\right)^{-1}\frac{d\varepsilon}{\varepsilon}.$$

We want to provide for $F(\xi',\delta)$ an estimate from below of the type considered in **h1**. To do so, for $|\xi'|\ge 1$ we split the integral above as

$$F(\xi',\delta)=\textstyle\int_0^{\frac{\sigma}{2|\xi'|}}+\int_{\frac{\sigma}{2|\xi'|}}^{\frac{\sigma}{|\xi'|}}+\int_{\frac{\sigma}{|\xi'|}}^1\,,$$

where $0 < \sigma < 1$ will be chosen later on. From the above decomposition it follows that

$$\begin{aligned} F(\xi',\delta) &\geq \int_{\frac{\sigma}{2|\xi'|}}^{\frac{\sigma}{|\xi'|}} |\widehat{\chi}(-\tfrac{i\varepsilon}{2},\varepsilon\xi')|^2\varepsilon^{-2k}\left(1+\tfrac{\delta^2}{\varepsilon^2}\right)^{-1}\tfrac{d\varepsilon}{\varepsilon} \\ &\geq |\xi'|^{2k}(1+|\tfrac{2\delta\xi'}{\sigma}|^2)^{-1}\int_{\frac{\sigma}{2|\xi'|}}^{\frac{\sigma}{|\xi'|}} |\widehat{\chi}(-\tfrac{i\varepsilon}{2},\varepsilon\xi')|^2\tfrac{\varepsilon^{-2k}}{|\xi'|^{2k}}\tfrac{d\varepsilon}{\varepsilon} \\ &= |\xi'|^{2k}(1+|\tfrac{2\delta\xi'}{\sigma}|^2)^{-1}\int_{\frac{\sigma}{2}}^{\sigma} |\widehat{\chi}(-\tfrac{it}{2|\xi'|},t\eta)|^2 t^{-2k}\tfrac{dt}{t}\,, \end{aligned}$$

where, in the last line, we have used the change of variable $t = \varepsilon|\xi'|$ and we have set $\eta = \frac{\xi'}{|\xi'|}$.
Thanks to the last inequality, in order to prove **h1**, it is sufficient to prove the following statement

h. There exist $\sigma \in]0,1[$, $M \geq 1$ and $C > 0$, such that for all $\xi' \in \mathbb{R}^n$ with $|\xi'| \geq M$

$$I(\xi') := \int_{\frac{\sigma}{2}}^{\sigma} |\widehat{\chi}(-\frac{it}{2|\xi'|},t\eta)|^2 t^{-2k}\frac{dt}{t} \geq C. \tag{118}$$

Note that by the trivial inequality

$$|\widehat{\chi}(-\frac{it}{2|\xi'|},t\eta)| \geq |\widehat{\chi}(0,t\eta)| - |\widehat{\chi}(-\frac{it}{2|\xi'|},t\eta) - \widehat{\chi}(0,t\eta)|,$$

to prove **h** it is sufficient to prove that the following two claims are true:

Claim 1 There exist $\sigma \in]0,1[$ and $C_2 > 0$ such that

$$|\widehat{\chi}(0,t\eta)| \geq C_2\,, \quad \forall\, t \in [\sigma/2,\sigma]\,,\ \forall \eta \in \mathbb{R}^n,\ \ |\eta| = 1\,. \tag{119}$$

Claim 2 There exists $M \geq 1$ such that for all $\xi' \in \mathbb{R}^n$ with $|\xi'| \geq M$ and all $t \in [\sigma/2,\sigma]$ then

$$|\widehat{\chi}(-\frac{it}{2|\xi'|},t\eta) - \widehat{\chi}(0,t\eta)| \leq \frac{C_2}{2}\,,$$

with the same C_2 involved in **Claim 1**.

In fact, from Claims 1 and 2, we get

$$|\widehat{\chi}(-\frac{it}{2|\xi'|},t\eta)| \geq |\widehat{\chi}(0,t\eta)| - |\widehat{\chi}(-\frac{it}{2|\xi'|},t\eta) - \widehat{\chi}(0,t\eta)| \geq C_2 - \frac{C_2}{2} = \frac{C_2}{2}\,,$$

for all $\xi' \in \mathbb{R}^n$ with $|\xi'| \geq M$ and for all $t \in [\sigma/2,\sigma]$. The last estimate implies the statement in **h**, provided that σ and M are the ones coming from Claims 1 and 2, respectively. To conclude, it remains to prove Claims 1 and 2.

Proof of Claim 1. We use the explicit form of χ given by (38) to get $\widehat{\chi}(\zeta) = |\zeta|^p\widehat{\phi}(\zeta)$, for $\zeta \in \mathbb{C}^{n+1}$. From $\widehat{\phi}(0) \neq 0$ and the continuity of $\widehat{\phi}$ we obtain

$$\exists\, C_1 > 0\,,\ \exists\, \sigma \in]0,1[:\ |\widehat{\phi}(0,t\eta)| \geq C_1\,, \quad \forall\, t \in]0,\sigma]\,,\ \forall \eta \in \mathbb{R}^n,\ \ |\eta| = 1\,. \tag{120}$$

Then (120) implies

$$\exists\, C_2 > 0\ :\ |\widehat{\chi}(0,t\eta)| \geq C_2\,, \quad \forall\, t \in [\sigma/2,\sigma]\,,\ \forall \eta \in \mathbb{R}^n,\ \ |\eta| = 1\,,$$

with the same σ as in (120), which concludes the proof of Claim 1.

Proof of Claim 2. We first observe that, with the choice of σ coming from (120), for $t \in [\sigma/2,\sigma]$ and $|\xi'| \geq 1$ we find that the points $(-\frac{it}{2|\xi'|},t\eta)$, $(0,t\eta)$ belong to the closed ball $B_{\sigma\sqrt{5}/2} := \{\zeta \in \mathbb{C}^{n+1}\ :\ |\zeta| \leq \sigma\sqrt{5}/2\}$.
Since $\widehat{\chi}$ is uniformly continuous on $B_{\sigma\sqrt{5}/2}$, there exists a positive σ_2 such that

$$|\widehat{\chi}(-\frac{it}{2|\xi'|},t\eta) - \widehat{\chi}(0,t\eta)| \leq \frac{C_2}{2}, \tag{121}$$

is satisfied as long as

$$\left|(-\frac{it}{2|\xi'|},t\eta) - (0,t\eta)\right| = \frac{t}{2|\xi'|} \leq \sigma_2. \tag{122}$$

In order to conclude, it remains to prove that there exists $M \geq 1$ such that for all $\xi' \in \mathbb{R}^n$ with $|\xi'| \geq M$, (122) holds for all $t \in [\sigma/2,\sigma]$; it can be proven that such an M exists and is given by $M = \max\{1, \frac{\sigma}{2\sigma_2}\}$.

This completes the proof of Proposition 12.

Appendix B. Properties of anisotropic Sobolev spaces

This section is devoted to some fundamental properties of function spaces H^m_* and H^m_{**}. We improve significantly known results in the literature [20, 29, 34] about imbeddings and products of functions, which are the fundamental tools for calculus in spaces H^m_*. In the sequel, we denote by $C^\infty_{(0)}(\Omega)$ the set of restrictions to Ω of functions of $C^\infty_0(\mathbb{R}^n)$. We first recall the following result.

Theorem 31. *For all integers $m \geq 1$, $C^\infty_{(0)}(\Omega)$ is dense in $H^m_*(\Omega)$ and dense in $H^m_{**}(\Omega)$.*

Proof. See [20]. □

We have the following estimates of Gagliardo-Nirenberg type.

Lemma 32. *For all functions $u \in H^m_*(\Omega) \cap L^\infty(\Omega)$ one has*

$$\|Z^\alpha \partial^h_{x_1} u\|_{L^{2/\gamma}(\Omega)} \leq C\|u\|^{1-\gamma}_{L^\infty(\Omega)}\|u\|^\gamma_{H^m_*(\Omega)}, \tag{123}$$

where $\gamma = 1-(1-\frac{h}{[m/2]})(1-\frac{|\alpha|}{m-2h})$, and provided that $0 \leq h < [m/2]$, $0 \leq |\alpha| < m-2h$.

Proof. By introducing a suitable partition of unity and a local coordinate change we reduce to the case $\mathbb{R}^n_+ = \{x = (x_1, x')|x_1 > 0, x' \in \mathbb{R}^{n-1}\}$. Given $u \in H^m_*(\mathbb{R}^n_+)$, we extend it to the whole space as in [14] by setting $\tilde{u}(x_1, x') = u(x_1, x')$ for $x_1 > 0$ and $\tilde{u}(x_1, x') = \sum_{j=1}^m \alpha_j u(-jx_1, x')$ for $x_1 < 0$. We impose the continuity conditions of $\partial^k_{x_1}\tilde{u}$ at $x_1 = 0$ for $k = 0, \dots, m-1$, which yield $\sum_{j=1}^m \alpha_j(-j)^k = 1$ for $k = 0, \dots, m-1$. As this linear system has a Vandermonde matrix, it is invertible and the coefficients $\alpha_1, \dots, \alpha_m$ are defined in a unique way. The extension is a linear continuous operator from $H^m_*(\mathbb{R}^n_+)$ into $H^m_*(\mathbb{R}^n)$ (with an obvious definition of the last space). Moreover, it is such that $\|Z^\alpha\partial^k_{x_1}\tilde{u}\|_{L^p(\mathbb{R}^n)} \leq C\|Z^\alpha\partial^k_{x_1}u\|_{L^p(\mathbb{R}^n_+)}$ for all $|\alpha| + 2k \leq m$ and all $1 \leq p \leq \infty$. Therefore, from now on we may work in $\mathbb{R}^n$; we also write u instead of $\tilde{u}$.

Let $p \geq 2$. By integration over $\Omega = \mathbb{R}^n$ of $Z_j(u\,Z_ju\,|Z_ju|^{p-2})$ we obtain as in [10] the estimate

$$\|Z_ju\|^2_{L^p(\Omega)} \leq (p-1)\|u\|_{L^q(\Omega)}\left(\|Z_j^2u\|_{L^r(\Omega)} + \delta_{1j}\|Z_1u\|_{L^r(\Omega)}\right) \tag{124}$$

for all $j = 1, \dots, n$ and $2/p = 1/q + 1/r$. By multiple application of (124) we obtain (proof by induction)

$$\|Z^\alpha u\|_{L^p(\Omega)} \leq C\|u\|^{1-|\alpha|/m}_{L^q(\Omega)}\textstyle\sum_{1\leq|\beta|\leq m}\|Z^\beta u\|^{|\alpha|/m}_{L^r(\Omega)}, \tag{125}$$

for $1/p = (1-|\alpha|/m)1/q + (|\alpha|/m)\,1/r$, and $1 \leq |\alpha| < m$. It yields in particular

$$\|Z^\alpha u\|_{L^{2m/|\alpha|}(\Omega)} \leq C\|u\|^{1-|\alpha|/m}_{L^\infty(\Omega)}\textstyle\sum_{1\leq|\beta|\leq m}\|Z^\beta u\|^{|\alpha|/m}_{L^2(\Omega)}, \tag{126}$$

for $1 \leq |\alpha| < m$. Similarly we can obtain

$$\|\partial^h_{x_1}u\|_{L^{2[m/2]/h}(\Omega)} \leq C\|u\|^{1-h/[m/2]}_{L^\infty(\Omega)}\|\partial^{[m/2]}_{x_1}u\|^{h/[m/2]}_{L^2(\Omega)}, \tag{127}$$

for $1 \leq h < [m/2]$. Combining (125), (126) and (127) gives

$$\begin{aligned}\|Z^\alpha\partial^h_{x_1}u\|_{L^p(\Omega)} &\leq C\|\partial^h_{x_1}u\|^{1-|\alpha|/(m-2h)}_{L^q(\Omega)}\textstyle\sum_{1\leq|\delta|\leq m-2h}\|Z^\delta\partial^h_{x_1}u\|^{|\alpha|/(m-2h)}_{L^2(\Omega)}\\ &\leq C\left(\|u\|^{1-h/[m/2]}_{L^\infty(\Omega)}\|\partial^{[m/2]}_{x_1}u\|^{h/[m/2]}_{L^2(\Omega)}\right)^{1-|\alpha|/(m-2h)}\textstyle\sum_{1\leq|\delta|\leq m-2h}\|Z^\delta\partial^h_{x_1}u\|^{|\alpha|/(m-2h)}_{L^2(\Omega)}\end{aligned} \tag{128}$$

where $1/p = (1-|\alpha|/(m-2h))1/q + (|\alpha|/(m-2h))\,1/2$, $1/q = (h/[m/2])1/2$ and $1 \leq |\alpha| < m-2h$, $1 \leq h < [m/2]$. From (128) we readily obtain (123). □

Let us define the space

$$W^{1,\infty}_*(\Omega) = \{u \in L^\infty(\Omega)\,:\,Z_iu \in L^\infty(\Omega),\, i = 1\dots,n\},$$

equipped with its natural norm. From Lemma 32 we deduce the following Moser-type inequalities.

Theorem 33. *Let $m \in \mathbb{N}$, $m \geq 1$. If m is 1 or even, for all functions u and v in $H^m_*(\Omega) \cap L^\infty(\Omega)$ one has*

$$\|uv\|_{H^m_*(\Omega)} \leq C(\|u\|_{H^m_*(\Omega)}\|v\|_{L^\infty(\Omega)} + \|u\|_{L^\infty(\Omega)}\|v\|_{H^m_*(\Omega)}). \tag{129}$$

If $m \geq 3$ is odd, for all functions u and v in $H^m_*(\Omega) \cap W^{1,\infty}_*(\Omega)$ one has

$$\|uv\|_{H^m_*(\Omega)} \leq C(\|u\|_{H^m_*(\Omega)}\|v\|_{W^{1,\infty}_*(\Omega)} + \|u\|_{W^{1,\infty}_*(\Omega)}\|v\|_{H^m_*(\Omega)}). \tag{130}$$

Proof. If $m = 1$ the proof is obvious, so we consider the case $m \geq 2$. Again, by localization we reduce to the case $\Omega = \mathbb{R}^n_+ = \{x = (x_1, x')|x_1 > 0, x' \in \mathbb{R}^{n-1}\}$. Assume first that m is even. By Leibniz's rule we have

$$\|uv\|_{H^m_*(\Omega)} \leq C \sum_{|\alpha|+2k\leq m} \sum_{\beta\leq\alpha,h\leq k} \|Z^{\alpha-\beta}\partial^{k-h}_{x_1}u\, Z^\beta\partial^h_{x_1}v\|_{L^2(\Omega)} = I_1 + I_2,$$

where we have denoted

$$I_1 = C \sum_{|\alpha|+2k\leq m} \left(\|v\, Z^\alpha\partial^k_{x_1}u\|_{L^2(\Omega)} + \|u\, Z^\alpha\partial^k_{x_1}v\|_{L^2(\Omega)}\right),$$

$$I_2 = C \sum_{|\alpha|+2k\leq m} \sum_{(\beta,h)\in K_1(\alpha,k)} \|Z^{\alpha-\beta}\partial^{k-h}_{x_1}u\, Z^\beta\partial^h_{x_1}v\|_{L^2(\Omega)},$$

$$K_1(\alpha, k) = \{(\beta, h) : \beta \leq \alpha,\, h \leq k,\, 1 \leq |\beta| + 2h \leq m - 1\}.$$

It is clear that I_1 may be readily estimated by the right-hand side of (129). As for I_2, from the Hölder's inequality we get

$$\|Z^{\alpha-\beta}\partial^{k-h}_{x_1}u\, Z^\beta\partial^h_{x_1}v\|_{L^2(\Omega)} \leq C\|Z^{\alpha-\beta}\partial^{k-h}_{x_1}u\|_{L^{2/\gamma}(\Omega)}\|Z^\beta\partial^h_{x_1}v\|_{L^{2/\delta}(\Omega)} \tag{131}$$

where γ and δ must satisfy $\gamma + \delta = 1$, when $|\alpha| + 2k = m$. In view of (123) we choose $\gamma = 1 - (1 - \frac{k-h}{[m/2]})(1 - \frac{|\alpha-\beta|}{m-2(k-h)})$, $\delta = 1 - (1 - \frac{h}{[m/2]})(1 - \frac{|\beta|}{m-2h})$. Here we notice that for the indices in I_2 one has $k - h < [m/2]$, $h < [m/2]$, $|\alpha - \beta| < m - 2(k - h)$, $|\beta| < m - 2h$. This choice of γ and δ enjoys the above requirement and yields, by applying (123),

$$\begin{aligned}\|Z^{\alpha-\beta}\partial^{k-h}_{x_1}u\, Z^\beta\partial^h_{x_1}v\|_{L^2(\Omega)} &\leq C\|u\|^{1-\gamma}_{L^\infty(\Omega)}\|u\|^\gamma_{H^m_*(\Omega)}\|v\|^\gamma_{L^\infty(\Omega)}\|v\|^{1-\gamma}_{H^m_*(\Omega)} \\ &\leq C(\|u\|_{L^\infty(\Omega)}\|v\|_{H^m_*(\Omega)} + \|u\|_{H^m_*(\Omega)}\|v\|_{L^\infty(\Omega)}).\end{aligned} \tag{132}$$

Adding over α, β and k, h completes the proof of (129). Assume now that $m \geq 3$ is odd. By Leibniz's rule we have

$$\|uv\|_{H^m_*(\Omega)} \leq C \sum_{|\alpha|+2k\leq m} \sum_{\beta\leq\alpha,h\leq k} \|Z^{\alpha-\beta}\partial^{k-h}_{x_1}u\, Z^\beta\partial^h_{x_1}v\|_{L^2(\Omega)} = J_1 + J_2 + J_3,$$

where we have denoted

$$J_1 = C \sum_{|\alpha|+2k\leq m} \left(\|v\, Z^\alpha\partial^k_{x_1}u\|_{L^2(\Omega)} + \|u\, Z^\alpha\partial^k_{x_1}v\|_{L^2(\Omega)}\right),$$

$$J_2 = C \sum_{|\alpha|\leq m} \sum_{\beta\in K_2(\alpha)} \|Z^{\alpha-\beta}u\, Z^\beta v\|_{L^2(\Omega)},$$

$$J_3 = C \sum_{|\alpha|+2k\leq m, k\geq 1} \sum_{(\beta,h)\in K_3(\alpha,k)} \|Z^{\alpha-\beta}\partial^{k-h}_{x_1}u\, Z^\beta\partial^h_{x_1}v\|_{L^2(\Omega)},$$

$$K_2(\alpha) = \{\beta : \beta \leq \alpha,\, 1 \leq |\beta| \leq m - 1\},$$

$$K_3(\alpha, k) = \{(\beta, h) : \beta \leq \alpha,\, h \leq k,\, 1 \leq |\alpha - \beta| + 2(k - h),\, 1 \leq |\beta| + 2h\}.$$

Again, J_1 may be readily estimated by the right-hand side of (129), and consequently by the right-hand side of (130). As for J_2, we can use the interpolating inequality (123) for purely tangential derivatives, proceed as in the proof for I_2, and get an estimate of J_2 by the right-hand side of (129).

Finally, let us consider J_3. If $|\alpha| + 2k < m$ we may apply (129) with $m - 1$ (which is even) instead of m. Let $|\alpha| + 2k = m$. Since m is odd, $|\alpha|$ is also odd. Then the generic term has either the form

$$\|Z^{\gamma-\beta}\partial^{k-h}_{x_1}Zu\, Z^\beta\partial^h_{x_1}v\|_{L^2(\Omega)} \tag{133}$$

if $|\beta|$ is even, where γ is such that $|\alpha| = |\gamma| + 1$, so that also $|\gamma - \beta|$ is even, or the form

$$\|Z^{\alpha-\beta}\partial_{x_1}^{k-h} u\, Z^{\delta}\partial_{x_1}^{h} Zv\|_{L^2(\Omega)} \tag{134}$$

if $|\beta|$ is odd, where δ is such that $|\beta| = |\delta| + 1$, so that $|\alpha - \beta|$ and $|\delta|$ are both even.

If $|\beta|$ is even and $|\beta| + 2h = m - 1$ then $|\gamma - \beta| + 2(k - h) = 0$ and the norm in (133) reduces to

$$\|Zu\, Z^{\beta}\partial_{x_1}^{h} v\|_{L^2(\Omega)} \le C\|Zu\|_{L^\infty(\Omega)}\|v\|_{H_*^{m-1}(\Omega)}. \tag{135}$$

Otherwise we have $|\beta| + 2h < m - 1$ and we may check that $|\gamma - \beta| + 2(k - h) < m - 1$, $k - h < [(m-1)/2]$, $h < [(m-1)/2]$. Then the norm in (133) may be estimated as we did for I_2, by applying the interpolating inequalities (123) in $H_*^{m-1}(\Omega)$ to the functions Zu and v. The norm in (133) is estimated by

$$C(\|Zu\|_{L^\infty(\Omega)}\|v\|_{H_*^{m-1}(\Omega)} + \|v\|_{L^\infty(\Omega)}\|u\|_{H_*^{m}(\Omega)}). \tag{136}$$

If $|\beta|$ is odd and $|\alpha - \beta| + 2(k - h) = m - 1$ then $|\beta| + 2h = 1$ and the norm in (134) reduces to

$$\|Z^{\alpha-\beta}\partial_{x_1}^{k-h} u\, Zv\|_{L^2(\Omega)} \le C\|u\|_{H_*^{m-1}(\Omega)}\|Zv\|_{L^\infty(\Omega)}. \tag{137}$$

Otherwise we have $|\alpha - \beta| + 2(k - h) < m - 1$ and we may check that $k - h < [(m-1)/2]$, $|\delta| + 2h < m-1$, $h < [(m-1)/2]$. In this case the norm in (134) may be estimated again by applying the interpolating inequalities (123) in $H_*^{m-1}(\Omega)$ to the functions u and Zv. The norm in (134) is estimated by

$$C(\|u\|_{H_*^{m-1}(\Omega)}\|Zv\|_{L^\infty(\Omega)} + \|u\|_{L^\infty(\Omega)}\|v\|_{H_*^{m}(\Omega)}). \tag{138}$$

By (135), (136) and (137), (138) we conclude the estimate of J_3. The proof of (130) is complete. □

Imbedding theorems for the anisotropic spaces $H_*^m(\Omega)$ follow in natural way from the inclusion $H_*^m(\Omega) \hookrightarrow H^{[m/2]}(\Omega)$ and the imbedding theorems for standard Sobolev spaces, see [20, 29, 34]. In particular, following this way one has the continuous imbedding $H_*^m(\Omega) \hookrightarrow C^0(\overline{\Omega})$ if m is such that $[m/2] > n/2$. This result is improved by the following theorem.

Theorem 34. *Let Ω be a bounded open subset of $\mathbb{R}^n$, $n \ge 2$, with C^∞ boundary. Then the continuous imbedding $H_*^{[(n+1)/2]+1}(\Omega) \hookrightarrow C^0(\overline{\Omega})$ holds.*

For example, in dimensions 2 and 3 we improve from $H_*^4(\Omega) \hookrightarrow H^2(\Omega) \hookrightarrow C^0(\overline{\Omega})$ to $H_*^2(\Omega) \hookrightarrow C^0(\overline{\Omega})$ and $H_*^3(\Omega) \hookrightarrow C^0(\overline{\Omega})$, respectively. We also observe that, for n even, the result improves the standard imbedding $H^{[n/2]+1}(\Omega) \hookrightarrow C^0(\overline{\Omega})$ to $H_*^{[n/2]+1}(\Omega) \hookrightarrow C^0(\overline{\Omega})$.

Proof. We observe that $C^\infty(\overline{\Omega})$ is dense in $H_*^m(\Omega)$ from Lemma 31. Thus it is sufficient to prove that

$$||u||_{L^\infty(\Omega)} \le C||u||_{H_*^m(\Omega)},$$

for all $u \in C^\infty(\overline{\Omega})$, where $m = [(n+1)/2] + 1$. By introducing a suitable partition of unity and local coordinate change we reduce to the case $\Omega = \mathbb{R}^n_+ = \{x = (x_1, x')|x_1 > 0, x' \in \mathbb{R}^{n-1}\}$, $u \in C^\infty_{(0)}(\mathbb{R}^n_+)$. We notice that $m = [n/2] + 1$ if n is even, while $m = [n/2] + 2$ if n is odd. For any x_1 we have

$$||u(x_1, \cdot)||^2_{H^{m-1}(\mathbb{R}^{n-1})} = (2\pi)^{-n+1}\int_{\mathbb{R}^{n-1}} | < \xi' >^{m-1} \hat{u}(x_1, \xi')|^2 d\xi' \tag{139}$$

where $\hat{u}$ denotes the partial Fourier transform of u defined by

$$\hat{u}(x_1, \xi') = \int_{\mathbb{R}^{n-1}} e^{-ix'\cdot\xi'} u(x_1, x')dx'$$

and where $< \xi' >= \sqrt{1+|\xi'|^2}$. The right-hand side of (139) is equal to ($\Re$ = real part)

$$\begin{aligned}
&-(2\pi)^{-n+1}\int_{\mathbb{R}^{n-1}} < \xi' >^{2m-2} (\int_{x_1}^{\infty} \partial_1|\hat{u}(\xi_1,\xi')|^2 d\xi_1)\, d\xi' \\
&= -(2\pi)^{-n+1}\int_{\mathbb{R}^{n-1}} < \xi' >^{2m-2} 2\Re(\int_{x_1}^{\infty} \hat{u}(\xi)\partial_1\overline{\hat{u}(\xi)}\, d\xi_1)\, d\xi' \\
&= -2(2\pi)^{-n+1}\Re\int_{x_1}^{+\infty}\int_{\mathbb{R}^{n-1}} < \xi' >^{m} \hat{u}(\xi) \times < \xi' >^{m-2} \partial_1\overline{\hat{u}(\xi)}\, d\xi \\
&\le 2||u||^2_{H^m_*(\mathbb{R}^n_+)},
\end{aligned}$$

by application of the Cauchy-Schwarz inequality. Thus we have

$$||u(x_1,\cdot)||_{H^{m-1}(\mathbb{R}^{n-1})} \le \sqrt{2}||u||_{H^m_*(\mathbb{R}^n_+)}. \tag{140}$$

From the definition of m we infer $2(m-1) > n-1$, which yields the continuous imbedding $H^{m-1}(\mathbb{R}^{n-1}) \hookrightarrow C^0_B(\mathbb{R}^{n-1})$. From (140) and this last inclusion we obtain the thesis. □

Considering the space $H^m_{**}(\Omega)$ gives a further improvement. Observe that the following criterion on m, $m > n/2$, is the same of standard Sobolev spaces. The imbedding $H^m_{**}(\Omega) \hookrightarrow C^0(\overline{\Omega})$ was previously known only for $m > 2[n/2]+1$, see [34].

Theorem 35. *Let Ω be a bounded open subset of $\mathbb{R}^n$, $n \ge 2$, with C^∞ boundary. If $m > n/2$ then $H^m_{**}(\Omega) \hookrightarrow C^0(\overline{\Omega})$.*

Proof. We proceed as in the proof of Theorem 34. For any x_1 we find

$$\begin{aligned}
&||u(x_1,\cdot)||^2_{H^{m-1/2}(\mathbb{R}^{n-1})} = -2(2\pi)^{-n+1}\Re\int_{x_1}^{+\infty}\int_{\mathbb{R}^{n-1}} < \xi' >^{m} \hat{u}(\xi) \times < \xi' >^{m-1} \partial_1\overline{\hat{u}(\xi)}\, d\xi \\
&\le 2||u||^2_{H^m_{**}(\mathbb{R}^n_+)},
\end{aligned} \tag{141}$$

by application of the Cauchy-Schwarz inequality. If $m > n/2$ we infer $2(m-1/2) > n-1$, which yields the continuous imbedding $H^{m-1/2}(\mathbb{R}^{n-1}) \hookrightarrow C^0_B(\mathbb{R}^{n-1})$. From (141) and this last inclusion we obtain the thesis. □

From Theorems 33 and 34 we immediately deduce the following theorem about products of functions.

Theorem 36. *Let $m \ge \big[(n+1)/2\big]+2$. Consider functions $u \in H^m_*(\Omega)$ and $v \in H^m_*(\Omega)$. Then $uv \in H^m_*(\Omega)$ and*

$$\|uv\|_{H^m_*(\Omega)} \le c\|u\|_{H^m_*(\Omega)}\|v\|_{H^m_*(\Omega)}.$$

The following theorem provides some summability properties of anisotropic Sobolev functions with low order of smoothness.

Theorem 37. *Let Ω be a bounded open subset of $\mathbb{R}^n$, where $n \ge 4$, with C^∞ boundary. Then the following continuous imbeddings hold true:*

a. *If $n > 5$ and $2 \le m < \frac{n-1}{2}$, then*

$$H^m_*(\Omega) \hookrightarrow L^r(\Omega)\,, \quad \forall r \in [2, r^*]\,, \quad \frac{1}{r^*} = \frac{1}{2} - \frac{m}{n+1}\,. \tag{142}$$

b. *If $n \ge 5$ is odd, then*

$$H_*^{\frac{n-1}{2}}(\Omega) \hookrightarrow L^r(\Omega)\,, \quad \forall\, r \in [2, n+1[\,. \tag{143}$$

c. *If $n \ge 4$ is even, then*

$$H_*^{\frac{n}{2}}(\Omega) \hookrightarrow L^r(\Omega)\,, \quad \forall\, r \in [2, 2n]\,. \tag{144}$$

Proof. As in the proof of Theorem 34, by introducing a partition of unity and performing local changes of coordinates, we may reduce to the case $\Omega = \mathbb{R}^n_+$. For $n > 5$, let m be a given integer such that $2 \le m < \frac{n-1}{2}$.

In order to prove the first imbedding (142), firstly we use the standard Sobolev imbedding

$$H^{m-1}(\mathbb{R}^{n-1}) \hookrightarrow L^q(\mathbb{R}^{n-1})\,, \tag{145}$$

where $\frac{1}{q} = \frac{1}{2} - \frac{m-1}{n-1} > 0$, to find that the inequality

$$||u(x_1,\cdot)||_{L^q(\mathbb{R}^{n-1})} \le C||u(x_1,\cdot)||_{H^{m-1}(\mathbb{R}^{n-1})}$$

holds for all $u \in C^\infty_{(0)}(\mathbb{R}^n_+)$ and a positive constant C independent of u. Then, estimating $||u(x_1,\cdot)||_{H^{m-1}(\mathbb{R}^{n-1})}$ as in the proof of Theorem 34 gives

$$||u||_{L^\infty(0,+\infty;\,L^q(\mathbb{R}^{n-1}))} \le C||u||_{H^m_*(\mathbb{R}^n_+)}\,. \tag{146}$$

Similarly, from the imbedding $H^m(\mathbb{R}^{n-1}) \hookrightarrow L^p(\mathbb{R}^{n-1})$, with $\frac{1}{p} = \frac{1}{2} - \frac{m}{n-1} > 0$, we derive

$$||u||^2_{L^2(0,+\infty;\,L^p(\mathbb{R}^{n-1}))} \le C\int_0^{+\infty} ||u(x_1,\cdot)||^2_{H^m(\mathbb{R}^{n-1})}dx_1 \le C||u||^2_{H^m_*(\mathbb{R}^n_+)}\,, \tag{147}$$

for a suitable $C > 0$ independent of u.

Let θ be arbitrarily fixed in $]0,1[$. For $\frac{1}{r} = \frac{\theta}{q} + \frac{1-\theta}{p}$ interpolation between $L^q(\mathbb{R}^{n-1})$ and $L^p(\mathbb{R}^{n-1})$ and Fubini's theorem gives

$$\begin{aligned} ||u||^r_{L^r(\mathbb{R}^n_+)} &= \textstyle\int_0^{+\infty} ||u(x_1,\cdot)||^r_{L^r(\mathbb{R}^{n-1})}dx_1 \le \int_0^{+\infty} ||u(x_1,\cdot)||^{r\theta}_{L^q(\mathbb{R}^{n-1})}||u(x_1,\cdot)||^{r(1-\theta)}_{L^p(\mathbb{R}^{n-1})}\,dx_1 \\ &\le ||u||^{r\theta}_{L^\infty(0,+\infty;\,L^q(\mathbb{R}^{n-1}))} \textstyle\int_0^{+\infty} ||u(x_1,\cdot)||^{r(1-\theta)}_{L^p(\mathbb{R}^{n-1})}\,dx_1\,. \end{aligned} \tag{148}$$

Setting $r(1-\theta) = 2$, from $\frac{1}{r} = \frac{\theta}{q} + \frac{1-\theta}{p}$ we compute for θ and r the values

$$\theta = \theta^* := \frac{\frac{1}{2} - \frac{1}{p}}{\frac{1}{2} - \frac{1}{p} + \frac{1}{q}} = \frac{2m}{n+1}\,, \qquad r = r^* := \frac{2(n+1)}{n+1-2m}.$$

Setting $r = r^*$ and $\theta = \theta^*$ in (148) and using estimates (146) and (147), we get

$$||u||_{L^{r^*}(\mathbb{R}^n_+)} \le C||u||_{H^m_*(\mathbb{R}^n_+)}\,, \tag{149}$$

which proves the imbedding $H^m_*(\mathbb{R}^n_+) \hookrightarrow L^{r^*}(\mathbb{R}^n_+)$. The imbedding $H^m_*(\mathbb{R}^n_+) \hookrightarrow L^r(\mathbb{R}^n_+)$ for all $r \in [2, r^*]$ immediately follows from the interpolation between $L^2(\mathbb{R}^n_+)$ and $L^{r^*}(\mathbb{R}^n_+)$. This ends the proof of (142).

Assume now that n is odd and ≥ 5. To prove the continuous imbedding (143), we apply again the inequality (146) for $m = \frac{n-1}{2}$ and $q = n-1$ (recall that (146) follows from (145) that is true if $m - 1 < \frac{n-1}{2}$). Then use the Sobolev imbedding

$$H^{\frac{n-1}{2}}(\mathbb{R}^{n-1}) \hookrightarrow L^p(\mathbb{R}^{n-1})\,, \quad \forall\, p \in [2,+\infty[\,,$$

to find

$$||u||^2_{L^2(0,+\infty;\,L^p(\mathbb{R}^{n-1}))} \le C||u||^2_{H_*^{\frac{n-1}{2}}(\mathbb{R}^n_+)}\,, \quad \forall\, p \in [2,+\infty[\,.$$

Arguing as before, we get for all $p \ge 2$

$$||u||^r_{L^r(\mathbb{R}^n_+)} \le ||u||^{r\theta}_{L^\infty(0,+\infty;\,L^q(\mathbb{R}^{n-1}))} \textstyle\int_0^{+\infty} ||u(x_1,\cdot)||^{r(1-\theta)}_{L^p(\mathbb{R}^{n-1})}\,dx_1\,, \tag{150}$$

where $\theta \in [0,1]$ and r are such that $r(1-\theta) = 2$ and $\frac{1}{r} = \frac{\theta}{q} + \frac{1-\theta}{p} = \frac{\theta}{n-1} + \frac{1-\theta}{p}$; hence we derive as before

$$||u||_{L^r(\mathbb{R}^n_+)} \le C||u||_{H_*^{\frac{n-1}{2}}(\mathbb{R}^n_+)} \tag{151}$$

where $r = 2(n-1)(\frac{1}{2} - \frac{1}{p} + \frac{1}{n-1})$. Then the continuous imbedding (143) follows by noticing that the function $p \mapsto r(p) = 2(n-1)(\frac{1}{2} - \frac{1}{p} + \frac{1}{n-1})$ is increasing and continuous over $[2,+\infty[$ and $r(p) \nearrow n+1$ as $p \nearrow +\infty$.

To conclude, we prove the continuous imbedding (144). Thus, we assume that $n \geq 4$ is even. Again, by (145) for $m = \frac{n}{2}$ we derive that

$$||u||_{L^\infty(0,+\infty;L^q(\mathbb{R}^{n-1}))} \leq C||u||_{H^m_*(\mathbb{R}^n_+)}\,,$$

where $\frac{1}{q} = \frac{1}{2} - \frac{m-1}{n-1} = \frac{1}{2(n-1)}$. Moreover, since $\frac{n}{2} > \frac{n-1}{2}$, the standard imbedding $H^{\frac{n}{2}}(\mathbb{R}^{n-1}) \hookrightarrow L^\infty(\mathbb{R}^{n-1})$ gives

$$||u||_{L^2(0,+\infty;L^\infty(\mathbb{R}^{n-1}))} \leq C||u||_{H^{\frac{n}{2}}_*(\mathbb{R}^n_+)}\,.$$

Then for all $r > q$ we find

$$||u(x_1,\cdot)||^r_{L^r(\mathbb{R}^{n-1})} \leq ||u(x_1,\cdot)||^{r-q}_{L^\infty(\mathbb{R}^{n-1})}||u(x_1,\cdot)||^q_{L^q(\mathbb{R}^{n-1})}\,.$$

Hence

$$||u||^r_{L^r(\mathbb{R}^n_+)} = \int_0^{+\infty} ||u(x_1,\cdot)||^r_{L^r(\mathbb{R}^{n-1})}\,dx_1 \leq ||u||^q_{L^\infty(0,+\infty;L^q(\mathbb{R}^{n-1}))} \int_0^{+\infty} ||u(x_1,\cdot)||^{r-q}_{L^\infty(\mathbb{R}^{n-1})}\,dx_1\,.$$

Setting now $r = q + 2 = 2n$ we derive the continuous imbedding $H^{\frac{n}{2}}_*(\mathbb{R}^n_+) \hookrightarrow L^{2n}(\mathbb{R}^n_+)$. Interpolation between $L^2(\mathbb{R}^n_+)$ and $L^{2n}(\mathbb{R}^n_+)$ gives the continuous imbeddings in (144).

□

The next theorem deals with the product of two anisotropic Sobolev functions, one of which has low order of smoothness.

Theorem 38. *Let $m \geq 1$ be an integer and $s = \max\left\{m, \left[\frac{n+1}{2}\right] + 2\right\}$. For $u \in H^m_*(\Omega)$ and $v \in H^s_*(\Omega)$ then $uv \in H^m_*(\Omega)$ and*

$$||uv||_{H^m_*(\Omega)} \leq c||u||_{H^m_*(\Omega)}||v||_{H^s_*(\Omega)}\,. \tag{152}$$

Proof. We suppose $\Omega = \mathbb{R}^n_+$; the general case can be reduced to this case by localization and flattening of the boundary.
If $m = s \geq \left[\frac{n+1}{2}\right] + 2$ then the result of Theorem 38 is just the one of Theorem 36. So, let us assume that $1 \leq m < s = \left[\frac{n+1}{2}\right] + 2$. For $m = 1$ the result is true as a consequence of the imbedding Theorem 34; indeed for $u \in H^1_*(\mathbb{R}^n_+)$ and $v \in H^{\left[\frac{n+1}{2}\right]+2}_*(\mathbb{R}^n_+)$, then $uv \in L^2(\mathbb{R}^n_+)$. Moreover for all $j = 1,\dots,n$:

$$Z_j(uv) = Z_j u\, v + u\, Z_j v$$

still belongs to $L^2(\mathbb{R}^n_+)$, as $Z_j v \in H^{\left[\frac{n+1}{2}\right]+1}_*(\mathbb{R}^n_+) \hookrightarrow L^\infty(\mathbb{R}^n_+)$ (again from Theorem 34). For $2 \leq m \leq s-1$, assume the result has been already proven up to the order $m-1$; we want to show that it is true up to m. Let us consider $u \in H^m_*(\mathbb{R}^n_+)$ and $v \in H^s_*(\mathbb{R}^n_+)$; by hypothesis, we know that $uv \in H^{m-1}_*(\mathbb{R}^n_+)$. It remains to prove that $Z^\alpha\partial_1^k(uv) \in L^2(\mathbb{R}^n_+)$, when $|\alpha| + 2k = m$. By Leibniz's formula, we compute

$$||Z^\alpha\partial_1^k(uv)||_{L^2(\mathbb{R}^n_+)} \leq C \sum_{(\beta,h),(\gamma,l)\in I(\alpha,k)} ||Z^\beta\partial_1^h u\, Z^\gamma\partial_1^l v||_{L^2(\mathbb{R}^n_+)}\,, \tag{153}$$

where $I(\alpha,k) := \{(\beta,h),(\gamma,l) :\ \beta+\gamma=\alpha\,,\ h+l=k\}$.

Let us assume $n \geq 4$. We split $I(\alpha,k)$ as $I(\alpha,k) = I_1(\alpha,k) \cup I_2(\alpha,k) \cup I_3(\alpha,k) \cup I_4(\alpha,k) \cup I_5(\alpha,k)$, where

$$\begin{aligned}
&I_1(\alpha,k) := \{(\beta,h),(\gamma,l) \in I(\alpha,k) :\ 2 \leq m-|\beta|-2h < \tfrac{n-1}{2} \text{ and } 2 \leq s-|\gamma|-2l < \tfrac{n-1}{2}\}\,;\\
&I_2(\alpha,k) := \{(\beta,h),(\gamma,l) \in I(\alpha,k) :\ m-|\beta|-2h \leq 1\}\,;\\
&I_3(\alpha,k) := \{(\beta,h),(\gamma,l) \in I(\alpha,k) :\ s-|\gamma|-2l \leq 1\}\,;\\
&I_4(\alpha,k) := \{(\beta,h),(\gamma,l) \in I(\alpha,k) :\ m-|\beta|-2h \geq \tfrac{n-1}{2}\}\,;\\
&I_5(\alpha,k) := \{(\beta,h),(\gamma,l) \in I(\alpha,k) :\ s-|\gamma|-2l \geq \tfrac{n-1}{2}\}
\end{aligned} \tag{154}$$

(notice that $I_1(\alpha,k)=\emptyset$, as long as $n\le 5$). According to the splitting above, we decompose the sum in the right-hand side of (153) as:

$$\sum_{(\beta,h),(\gamma,l)\in I(\alpha,k)} ||Z^\beta\partial_1^h u\, Z^\gamma\partial_1^l v||_{L^2(\mathbb{R}^n_+)} = K_1+K_2+K_3+K_4+K_5\,,$$

and we estimate separately each term, where for $i=1,\dots,5$

$$K_i := \sum_{(\beta,h),(\gamma,l)\in I_i(\alpha,k)} ||Z^\beta\partial_1^h u\, Z^\gamma\partial_1^l v||_{L^2(\mathbb{R}^n_+)}\,.$$

We consider K_1. From Theorem 37, a., we have for all $(\beta,h),(\gamma,l)\in I_1(\alpha,k)$:

$$\begin{aligned}
&Z^\beta\partial_1^h u\in H_*^{m-|\beta|-2h}(\mathbb{R}^n_+)\hookrightarrow L^2(\mathbb{R}^n_+)\cap L^p(\mathbb{R}^n_+)\,, \quad \tfrac{1}{p}=\tfrac12-\tfrac{m-|\beta|-2h}{n+1}\,;\\
&Z^\gamma\partial_1^l v\in H_*^{s-|\gamma|-2l}(\mathbb{R}^n_+)\hookrightarrow L^2(\mathbb{R}^n_+)\cap L^q(\mathbb{R}^n_+)\,, \quad \tfrac{1}{q}=\tfrac12-\tfrac{s-|\gamma|-2l}{n+1}\,.
\end{aligned}$$

Since $\frac1p+\frac1q=:\frac1s\le\frac12$, we get

$$\begin{aligned}
&||Z^\beta\partial_1^h u\, Z^\gamma\partial_1^l v||_{L^2(\mathbb{R}^n_+)}\le ||Z^\beta\partial_1^h u||_{L^{\frac{2p}{s}}(\mathbb{R}^n_+)}||Z^\gamma\partial_1^l v||_{L^{\frac{2q}{s}}(\mathbb{R}^n_+)}\\
&\le ||Z^\beta\partial_1^h u||^{1-\theta_1}_{L^2(\mathbb{R}^n_+)}||Z^\beta\partial_1^h u||^{\theta_1}_{L^p(\mathbb{R}^n_+)}||Z^\gamma\partial_1^l v||^{1-\theta_2}_{L^2(\mathbb{R}^n_+)}||Z^\gamma\partial_1^l v||^{\theta_2}_{L^q(\mathbb{R}^n_+)}\\
&\le ||Z^\beta\partial_1^h u||^{1-\theta_1}_{L^2(\mathbb{R}^n_+)}||Z^\beta\partial_1^h u||^{\theta_1}_{H_*^{m-|\beta|-2h}(\mathbb{R}^n_+)}||Z^\gamma\partial_1^l v||^{1-\theta_2}_{L^2(\mathbb{R}^n_+)}||Z^\gamma\partial_1^l v||^{\theta_2}_{H_*^{s-|\gamma|-2l}(\mathbb{R}^n_+)}\\
&\le ||u||_{H_*^m(\mathbb{R}^n_+)}||v||_{H_*^s(\mathbb{R}^n_+)}\,,
\end{aligned}$$

where $\theta_1:=\frac{p-s}{p-2}$ and $\theta_2=\frac{q-s}{q-2}$.

Let us consider now K_2. For all $(\beta,h),(\gamma,l)\in I_2(\alpha,k)$, one has $|\beta|+2h\ge m-1$ and $|\gamma|+2l=m-(|\beta|+2h)\le 1$. Then, $Z^\gamma\partial_1^l v\in H_*^{s-1}(\mathbb{R}^n_+)\hookrightarrow L^\infty(\mathbb{R}^n_+)$ by the imbedding Theorem 34. One immediately derives

$$||Z^\beta\partial_1^h u\, Z^\gamma\partial_1^l v||_{L^2(\mathbb{R}^n_+)}\le ||Z^\beta\partial_1^h u||_{L^2(\mathbb{R}^n_+)}||Z^\gamma\partial_1^l v||_{L^\infty(\mathbb{R}^n_+)}\le ||u||_{H_*^m(\mathbb{R}^n_+)}||v||_{H_*^s(\mathbb{R}^n_+)}\,.$$

Let us estimate K_3. Firstly, we observe that for $2\le m<s-1$, $I_3(\alpha,k)$ is empty, because $(\gamma,l)\in I_3(\alpha,k)$ would satisfy both $|\gamma|+2l\ge s-1$ and $|\gamma|+2l\le|\alpha|+2k=m$. For $m=s-1$, one computes that all $(\beta,h),(\gamma,l)\in I_3(\alpha,k)$ satisfy $|\gamma|+2l=m$ (thus $(\gamma,l)=(\alpha,k)$) and $|\beta|+2h=m-(|\gamma|+2l)=0$ (thus $(\beta,h)=(0,0)$). Again by Theorem 34 (applied to $H_*^m(\mathbb{R}^n_+)=H_*^{s-1}(\mathbb{R}^n_+)$), this yields

$$||Z^\beta\partial_1^h u\, Z^\gamma\partial_1^l v||_{L^2(\mathbb{R}^n_+)}=||u\, Z^\alpha\partial_1^k v||_{L^2(\mathbb{R}^n_+)}\le ||u||_{L^\infty(\mathbb{R}^n_+)}||Z^\alpha\partial_1^k v||_{L^2(\mathbb{R}^n_+)}\le ||u||_{H_*^m(\mathbb{R}^n_+)}||v||_{H_*^s(\mathbb{R}^n_+)}\,.$$

Let us consider now the term K_4. We divide the proof in several steps.

i) First, we assume that $n\ge 4$ is even. Setting $n=2k$ (k integer ≥ 2), we compute that $\frac{n-1}{2}=k-\frac12$ and $\left[\frac{n+1}{2}\right]+1=k+1$. Since each $(\beta,h)\in I_4(\alpha,k)$ satisfies $m-|\beta|-2h\ge\frac{n-1}{2}$ and $m\le\left[\frac{n+1}{2}\right]+1$, we deduce $k-\frac12\le m\le k+1$, hence $k\le m\le k+1$.

i.1) For $m=k$, inequality $m-|\beta|-2h\ge\frac{n-1}{2}$ implies that $|\beta|+2h=0$ and $|\gamma|+2l=m-|\beta|-2h=k$. Hence, by the use of Theorem 37, c., we obtain

$$Z^\beta\partial_1^h u=u\in H_*^m(\mathbb{R}^n_+)=H_*^k(\mathbb{R}^n_+)=H_*^{\frac n2}(\mathbb{R}^n_+)\hookrightarrow L^2(\mathbb{R}^n_+)\cap L^{2n}(\mathbb{R}^n_+)\hookrightarrow L^n(\mathbb{R}^n_+)$$

and, since $s-k=\left[\frac{n+1}{2}\right]+2-k=2$, the imbedding Theorem for ordinary Sobolev spaces gives

$$Z^\gamma\partial_1^l v\in H_*^{s-k}(\mathbb{R}^n_+)=H_*^2(\mathbb{R}^n_+)\hookrightarrow H^1(\mathbb{R}^n_+)\hookrightarrow L^{2^*}(\mathbb{R}^n_+)\,,\quad \frac{1}{2^*}=\frac12-\frac1n\,.$$

Hence, we obtain that

$$||Z^\beta\partial_1^h u\, Z^\gamma\partial_1^l v||_{L^2(\mathbb{R}^n_+)}=||u\, Z^\gamma\partial_1^l v||_{L^2(\mathbb{R}^n_+)}\le ||u||_{L^n(\mathbb{R}^n_+)}||Z^\gamma\partial_1^l v||_{L^{2^*}(\mathbb{R}^n_+)}\le ||u||_{H_*^m(\mathbb{R}^n_+)}||v||_{H_*^s(\mathbb{R}^n_+)}\,.$$

i.2) For $m=k+1$, we find that for $(\beta,h)\in I_4(\alpha,k)$, $m-|\beta|-2h=k+1-|\beta|-2h\ge k-\frac12$ implies $|\beta|+2h\le 1$. We have to consider two cases.

i.2.1) For $|\beta|+2h=0$, then $|\gamma|+2l=k+1$; hence by Theorem 34 we get $Z^\beta\partial_1^h u=u\in H^m_*(\mathbb{R}^n_+)=H_*^{\left[\frac{n+1}{2}\right]+1}(\mathbb{R}^n_+)\hookrightarrow L^\infty(\mathbb{R}^n_+)$, then

$$||Z^\beta\partial_1^h u\,Z^\gamma\partial_1^l v||_{L^2(\mathbb{R}^n_+)}\le||u||_{L^\infty(\mathbb{R}^n_+)}||Z^\gamma\partial_1^l v||_{L^2(\mathbb{R}^n_+)}\le||u||_{H^m_*(\mathbb{R}^n_+)}||v||_{H^s_*(\mathbb{R}^n_+)}\,.$$

i.2.2) For $|\beta|+2h=1$ we have $|\gamma|+2l=k$; then, as in step i.1), we find

$$Z^\beta\partial_1^h u\in H^k_*(\mathbb{R}^n_+)=H_*^{\frac{n}{2}}(\mathbb{R}^n_+)\hookrightarrow L^2(\mathbb{R}^n_+)\cap L^{2n}(\mathbb{R}^n_+)\hookrightarrow L^n(\mathbb{R}^n_+)\,,$$

and

$$Z^\gamma\partial_1^l v\in H^{s-k}_*(\mathbb{R}^n_+)=H^2_*(\mathbb{R}^n_+)\hookrightarrow H^1(\mathbb{R}^n_+)\hookrightarrow L^{2^*}(\mathbb{R}^n_+)\,,\quad \frac{1}{2^*}=\frac{1}{2}-\frac{1}{n}\,.$$

and we conclude as for i.1).
ii) Assume now that $n\ge 5$ is odd; setting $n=2k+1$ (k integer ≥ 2), we compute $\frac{n-1}{2}=k$ and $\left[\frac{n+1}{2}\right]+1=k+2$. Hence, from $m-|\beta|-2h\ge\frac{n-1}{2}$ and $m\le\left[\frac{n+1}{2}\right]+1$, we find that $k\le m\le k+2$. We have to consider three different cases.
ii.1) For $m=k$, inequalities $k\ge m-|\beta|-2h\ge\frac{n-1}{2}=k$ imply that $|\beta|+2h=0$ and $|\gamma|+2l=k$. Then, by Theorem 37, b., we get

$$Z^\beta\partial_1^h u=u\in H^k_*(\mathbb{R}^n_+)=H_*^{\frac{n-1}{2}}(\mathbb{R}^n_+)\hookrightarrow L^r(\mathbb{R}^n_+)\,,\quad\forall\, r\in[2,n+1[\tag{155}$$

and, since $s-k=\left[\frac{n+1}{2}\right]+2-k=3$,

$$Z^\gamma\partial_1^l v\in H^{s-k}_*(\mathbb{R}^n_+)=H^3_*(\mathbb{R}^n_+)\hookrightarrow H^1(\mathbb{R}^n_+)\hookrightarrow L^{2^*}(\mathbb{R}^n_+)\,,\quad \frac{1}{2^*}=\frac{1}{2}-\frac{1}{n}\,. \tag{156}$$

Using (155) with $r=n$ and (156), we conclude again as in step i.1).
ii.2) For $m=k+1$, inequality $k+1-|\beta|-2h\ge\frac{n-1}{2}=k$ gives that $|\beta|+2h\le 1$; moreover $k=m-1\le|\gamma|+2l\le m=k+1$. Since $s-(k+1)=\left[\frac{n+1}{2}\right]+1-k=2$, applying again Theorem 37, b., for $r=n$, and the ordinary Sobolev imbedding Theorem yield

$$\begin{aligned}&Z^\beta\partial_1^h u\in H^{m-1}_*(\mathbb{R}^n_+)=H^k_*(\mathbb{R}^n_+)=H_*^{\frac{n-1}{2}}(\mathbb{R}^n_+)\hookrightarrow L^n(\mathbb{R}^n_+)\,,\\&Z^\gamma\partial_1^l v\in H_*^{s-(k+1)}(\mathbb{R}^n_+)=H^2_*(\mathbb{R}^n_+)\hookrightarrow H^1(\mathbb{R}^n_+)\hookrightarrow L^{2^*}(\mathbb{R}^n_+)\,,\quad \tfrac{1}{2^*}=\tfrac{1}{2}-\tfrac{1}{n}\end{aligned}$$

and we conclude as in the preceding case.
ii.3) For $m=k+2$, inequality $k+2-|\beta|-2h\ge k$ implies $|\beta|+2h\le 2$. We consider two different cases.
ii.3.1) When $1\le|\beta|+2h\le 2$ then $k\le|\gamma|+2l\le k+1$ and $s-(k+1)=\left[\frac{n+1}{2}\right]+1-k=2$. Thus Theorem 37, b., and the standard Sobolev imbedding Theorem imply again

$$\begin{aligned}&Z^\beta\partial_1^h u\in H^{m-2}_*(\mathbb{R}^n_+)=H^k_*(\mathbb{R}^n_+)=H_*^{\frac{n-1}{2}}(\mathbb{R}^n_+)\hookrightarrow L^n(\mathbb{R}^n_+),\\&Z^\gamma\partial_1^l v\in H_*^{s-(k+1)}(\mathbb{R}^n_+)=H^2_*(\mathbb{R}^n_+)\hookrightarrow L^{2^*}(\mathbb{R}^n_+)\,,\quad \tfrac{1}{2^*}=\tfrac{1}{2}-\tfrac{1}{n}\end{aligned}$$

and we conclude as in the preceding case.
ii.3.2) When $|\beta|+2h=0$, Theorem 34 immediately yields

$$Z^\beta\partial_1^h u=u\in H^m_*(\mathbb{R}^n_+)=H^{k+2}_*(\mathbb{R}^n_+)=H_*^{\left[\frac{n+1}{2}\right]+1}(\mathbb{R}^n_+)\hookrightarrow L^\infty(\mathbb{R}^n_+)$$

and we conclude as in case i.2.1) before.
Gathering all of the estimates collected in cases i.1),$\cdots$, ii.3.2) above gives the desired estimate for K_4.

At last, the estimate of K_5 is deduced by similar arguments; therefore we omit it for shortness.
Gathering all of the estimates collected for each of the different terms $K_1,\dots,K_5$ before gives that the derivatives $Z^\alpha\partial_1^h(uv)\in L^2(\mathbb{R}^n_+)$, whenever $|\alpha|+2k=m$, then $uv\in H^m_*(\mathbb{R}^n_+)$. Combining the found estimates of $K_1,\dots,K_5$ with (153) gives (152).
The argument above requires the use of Theorem 37, hence the dimension n has to be strictly larger than 3. We need to treat the cases $n=2$ and $n=3$ separately.
Case $n=2$**:** in this case we compute $s=[3/2]+2=3$ and what we need to prove is just that $uv\in H^2_*(\mathbb{R}^2_+)$, whenever $u\in H^2_*(\mathbb{R}^2_+)$ and $v\in H^3_*(\mathbb{R}^2_+)$ (recall that the result of Theorem 38 is true when $m=1$, for all dimensions $n\ge 2$). Note that, for $n=2$, Theorem 34 gives the continuous imbedding

$H^2_*(\mathbb{R}^2_+) \hookrightarrow L^\infty(\mathbb{R}^2_+)$. In view of $H^1_*(\mathbb{R}^2_+) \cdot H^3_*(\mathbb{R}^2_+) \hookrightarrow H^1_*(\mathbb{R}^2_+)$, we already know that $uv \in H^1_*(\mathbb{R}^2_+)$. In order to check that $uv \in H^2_*(\mathbb{R}^2_+)$ we still need to show that

$$\partial_1(uv) \in L^2(\mathbb{R}^2_+) \quad \text{and} \quad Z_j(uv) \in H^1_*(\mathbb{R}^2_+).$$

Leibniz's formula gives $\partial_1(uv) = \partial_1 u\, v + u\, \partial_1 v$; hence $\partial_1(uv) \in L^2(\mathbb{R}^2_+)$, as $\partial_1 u, \partial_1 v \in L^2(\mathbb{R}^2_+)$ and $u, v \in H^2_*(\mathbb{R}^2_+) \hookrightarrow L^\infty(\mathbb{R}^2_+)$. As for the tangential derivatives $Z_j(uv)$, Leibniz's rule gives again $Z_j(uv) = Z_j u\, v + u\, Z_j v$. Applying another first order tangential derivative and using once more Leibniz's rule give also

$$Z_h(Z_j(uv)) = Z^2_{h,j} u\, v + Z_j u\, Z_h v + Z_h u\, Z_j v + u\, Z^2_{j,h} v \in L^2(\mathbb{R}^2_+)\,,$$

since again all of the different terms, involved in the right-hand side of the identity above, are products of a function in $L^2(\mathbb{R}^2_+)$ and a function in $L^\infty(\mathbb{R}^2_+)$ (because of the continuous imbedding $H^2_*(\mathbb{R}^2_+) \hookrightarrow L^\infty(\mathbb{R}^2_+)$). This proves that $Z_j(uv) \in H^1_*(\mathbb{R}^2_+)$ and completes the proof.

Case $n = 3$**:** The proof for the case $n = 3$ follows by similar arguments, by using the continuous imbedding $H^3_*(\mathbb{R}^3_+) \hookrightarrow L^\infty(\mathbb{R}^3_+)$ and standard imbeddings for the usual Sobolev spaces $H^m(\mathbb{R}^3_+)$. We omit it for shortness. This completes the proof. □

Finally, we give some lemmata useful in the proof of the main Theorem 2. The next lemma improves the result of [29, Lemma A.4] from $\sigma = 2[n/2] + 4$ to $\sigma = [(n+1)/2] + 3$.

Lemma 39. *Let* $\sigma \geq [(n+1)/2] + 3$ *and let* A *be a matrix-valued function such that* $A \in H^\sigma_*(\mathbb{R}^n_+)$ *and* $A = 0$ *if* $x_1 = 0$*. Then, for each regular enough vector-valued function* u

$$\|A\partial_1 u\|_{L^2(\mathbb{R}^n_+)} \leq c\|A\|_{H^\sigma_*(\mathbb{R}^n_+)} \|Z_1 u\|_{L^2(\mathbb{R}^n_+)}. \tag{157}$$

Proof. Let

$$H(x_1, x') = (x_1)^{-1} \int_0^{x_1} \partial_1 A(y, x')dy = A(x_1, x')/x_1.$$

Then, $A\partial_1 u = Hx_1\partial_1 u$. By Theorem 34 we infer

$$\|H\|_{L^\infty(\mathbb{R}^n_+)} \leq \|\partial_1 A\|_{L^\infty(\mathbb{R}^n_+)} \leq C\|\partial_1 A\|_{H^{[(n+1)/2]+1}_*(\mathbb{R}^n_+)} \leq C\|A\|_{H^{[(n+1)/2]+3}_*(\mathbb{R}^n_+)},$$

which gives (157). □

Lemma 40. *Let* $\sigma \geq 2$*. Let* $A \in H^\sigma_*(\mathbb{R}^n_+)$ *be a matrix-valued function such that* $A = 0$ *if* $x_1 = 0$ *and let* H *be defined as in the proof of Lemma 39. Then*

$$\|H\|_{H^{\sigma-2}_*(\mathbb{R}^n_+)} \leq c\|A\|_{H^\sigma_*(\mathbb{R}^n_+)}.$$

Appendix C. Study of the commutator $[L, Z_i]$

Let's study in more detail the regularity of the matrices $\Gamma_\beta, \Gamma_0, \Psi$ in (99).

Lemma 41. *Let* σ *be an integer such that* $\sigma \geq [(n+1)/2] + 4$*. Assume that* $A_j \in \mathcal{C}_T(H^\sigma_*)$*, for* $j = 1, \ldots, n$*,* $B \in \mathcal{C}_T(H^{\sigma-2}_*)$*. Then the matrices* $\Gamma_\beta, \Gamma_0, \Psi$ *of formula* (99) *satisfy*

$$\Gamma_\beta \in \mathcal{C}_T(H^{\sigma-3}_*), \quad \Gamma_0 \in \mathcal{C}_T(H^{\sigma-2}_*), \quad \Psi \in \mathcal{C}_T(H^{\sigma-1}_*).$$

Under the same assumption for A_j*, if* $B \in \mathcal{C}_T(H^{\sigma-1}_*)$ *then* $\Gamma_0 \in \mathcal{C}_T(H^{\sigma-1}_*)$.

Proof. We develop our analysis as in [22]. Consider first the case $i > 1$. Then $[A_1\partial_1, Z_i] = -(Z_i A_1)\partial_1$. Recalling the decomposition $A_1 = A^1_1 + A^2_1$ given in (76), under our assumptions the matrix H defined in (77) satisfies $H \in \mathcal{C}_T(H^{\sigma-2}_*)$. It follows that $(Z_i A^2_1)\partial_1 = (Z_i H)Z_1$ with coefficient $Z_i H \in \mathcal{C}_T(H^{\sigma-3}_*)$. This is the term giving the biggest loss of regularity. As for A^1_1 we have

$$(Z_i A^1_1)\partial_1 u = (Z_i A^{I,I}_1)\partial_1 u^I = -(Z_i A^{I,I}_1)(A^{I,I}_1)^{-1} \left(A^{I,II}_1 \partial_1 u^{II} + \left(\textstyle\sum_{j=2}^{n+1} A_j Z_j u + Bu - Lu \right)^I \right). \tag{158}$$

Since $A^{I,II}_1$ vanishes at $\{x_1 = 0\}$, we can write

$$-(Z_i A^{I,I}_1)(A^{I,I}_1)^{-1} A^{I,II}_1 \partial_1 u^{II} = H_2 Z_1 u,$$

with H_2 defined as in (77). Using Theorem 38 yields $H_2 \in \mathcal{C}_T(H_*^{\sigma-2})$. For the other terms in (158) the analysis is straightforward. If $i = 1$, we readily get the thesis from the equality

$$[A_1\partial_1, Z_1] = A_1\partial_1 - (Z_1A_1)\partial_1 = L - \sum_{j=2}^{n+1} A_jZ_j - B - (\partial_1A_1)Z_1.$$

The analysis for the other terms in L is similar. □

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