

## UN PO' DI TERMINOLOGIA

$$f: A \subseteq \mathbb{R} \longrightarrow \mathbb{R}$$

$$\downarrow \quad x \longmapsto f(x)$$

" $f$  associa  $f(x)$  ad  $x$ "

FUNZIONE REALE DI  
UNA VARIABILE REALE

$x$  : variabile  
indipendente

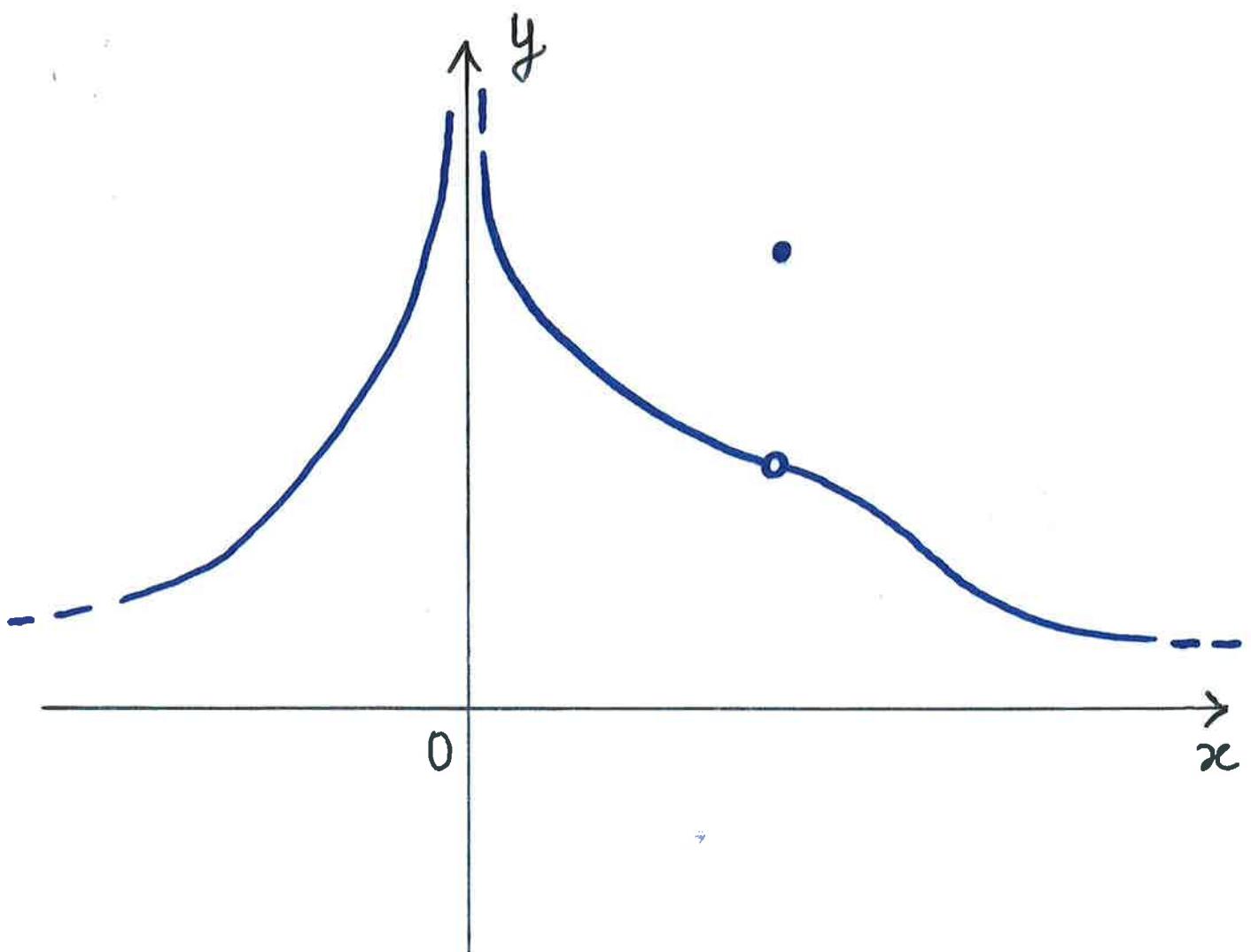
$y = f(x)$  : variabile  
dipendente

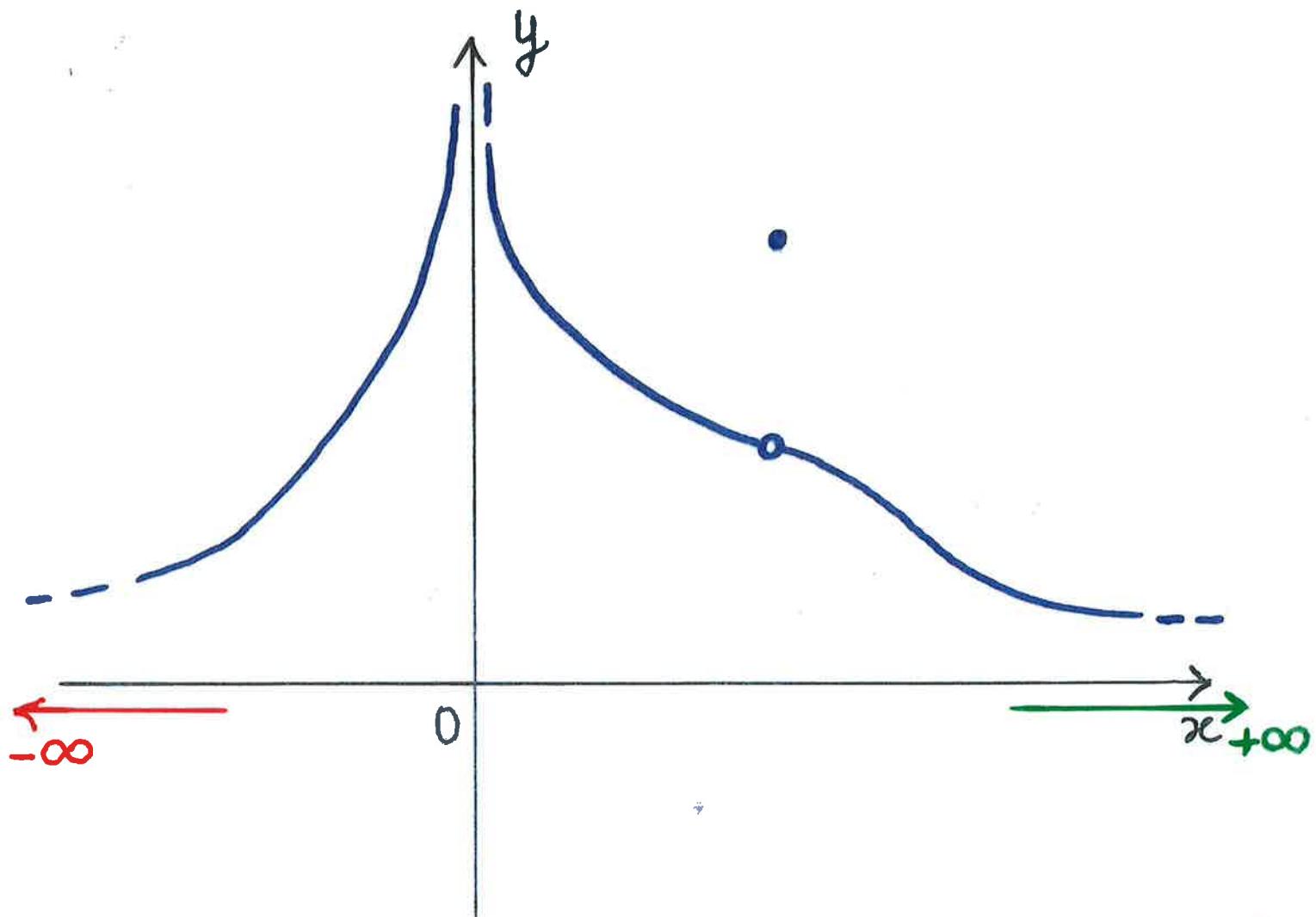
( $f(x)$  è il valore di  $f$   
in  $x$ )

- $A = \mathbb{N}$        $f: \mathbb{N} \longrightarrow \mathbb{R}$       SUCCESSIONE ( $\{a_m\}$ )  
 $(A = \{m \in \mathbb{N} : m \geq m_0\})$        $m \longmapsto f(m) = a_m$

$m$  : indice della  
successione

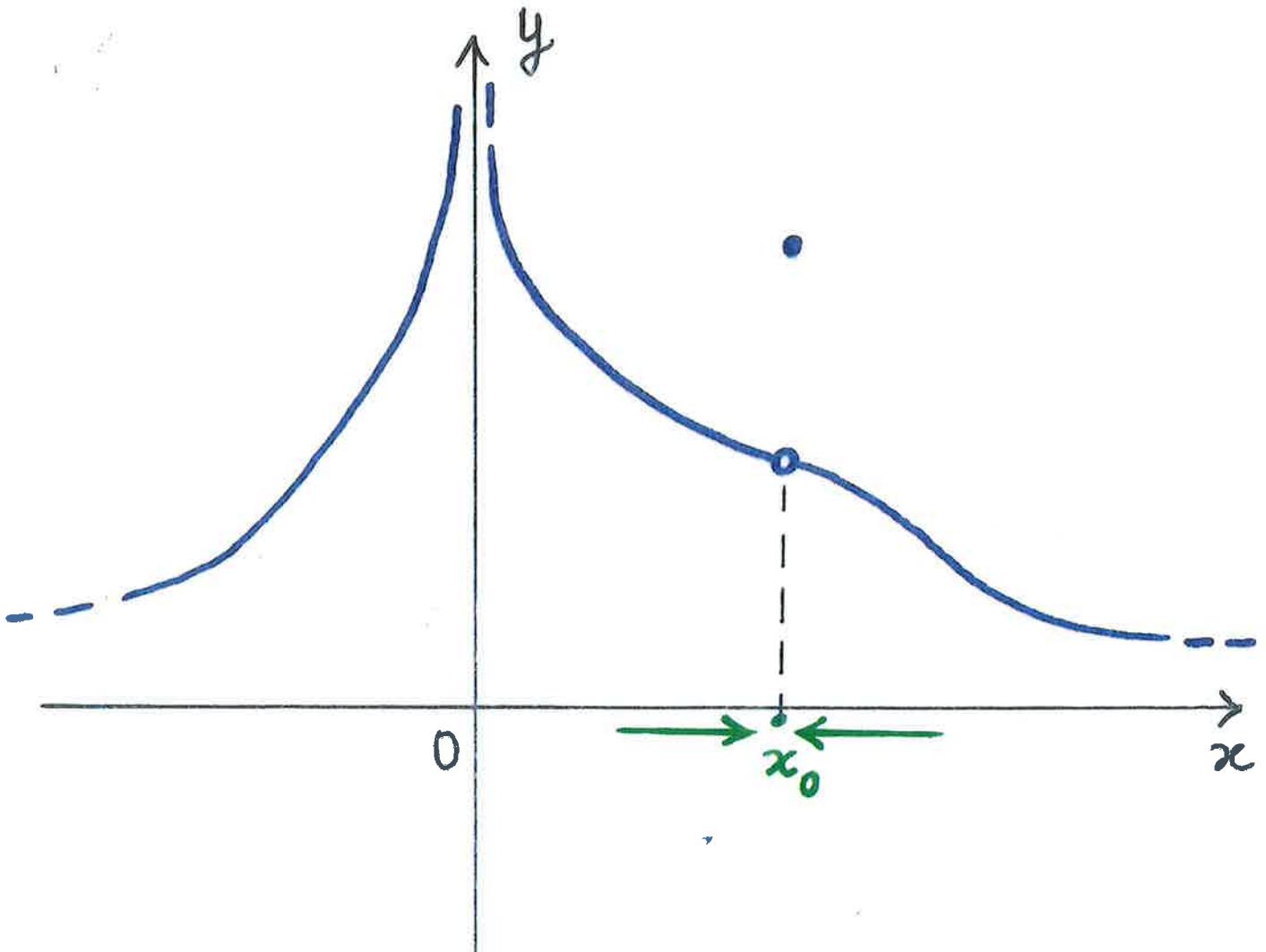
$a_m$  : valore  $m - m_0$   
della successione





$$\lim_{\substack{x \rightarrow +\infty \\ -\infty}} f(x)$$

LIMITE ALL' INFINITO



$$\lim_{x \rightarrow x_0} f(x)$$

LIMITE AL FINITO

## LIMITI DI FUNZIONI (REALI)

Estendiamo la nozione di limite all'infinito a funzioni reali di variabile reale.

**Def.** Sia  $f : \mathbf{R} \rightarrow \mathbf{R}$ ; diremo che  $f(x)$  tende al numero  $L \in \mathbf{R}$  per  $x \rightarrow +\infty$  se

$$\forall \varepsilon > 0 \exists M_{\varepsilon} > 0 : x \geq M_{\varepsilon} \implies |L - f(x)| \leq \varepsilon.$$

In tal caso il numero  $L$  si dice il limite di  $f$  per  $x \rightarrow +\infty$ , e si scrive  $\lim_{x \rightarrow +\infty} f(x) = L$ .

Allo stesso modo si definiscono

i)  $\lim_{x \rightarrow +\infty} f(x) = +\infty \iff$

$$\forall N > 0 \exists M_N > 0 : \forall x \geq M_N f(x) \geq N,$$

ii)  $\lim_{x \rightarrow +\infty} f(x) = -\infty \iff$

$$\forall N > 0 \exists M_N > 0 : \forall x \geq M_N f(x) \leq -N.$$

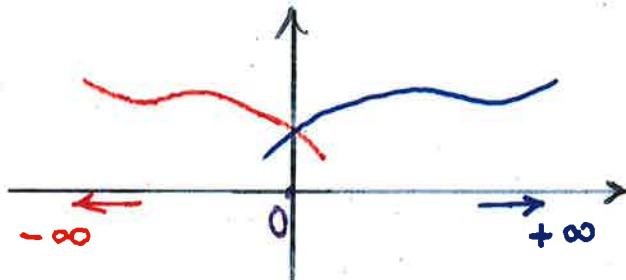
La nozione di limite per  $x \rightarrow -\infty$  viene data per simmetria:

\*  $\lim_{x \rightarrow -\infty} f(x) = L \iff$

$$\forall \varepsilon > 0 \exists M_{\varepsilon} > 0 : \forall x \leq -M_{\varepsilon} |f(x) - L| \leq \varepsilon,$$

ovvero  $\lim_{x \rightarrow -\infty} f(x) = L \iff \lim_{x \rightarrow +\infty} f(-x) = L$ .

Definiamo inoltre:



$$*\text{ i) } \lim_{x \rightarrow -\infty} f(x) = +\infty \iff$$

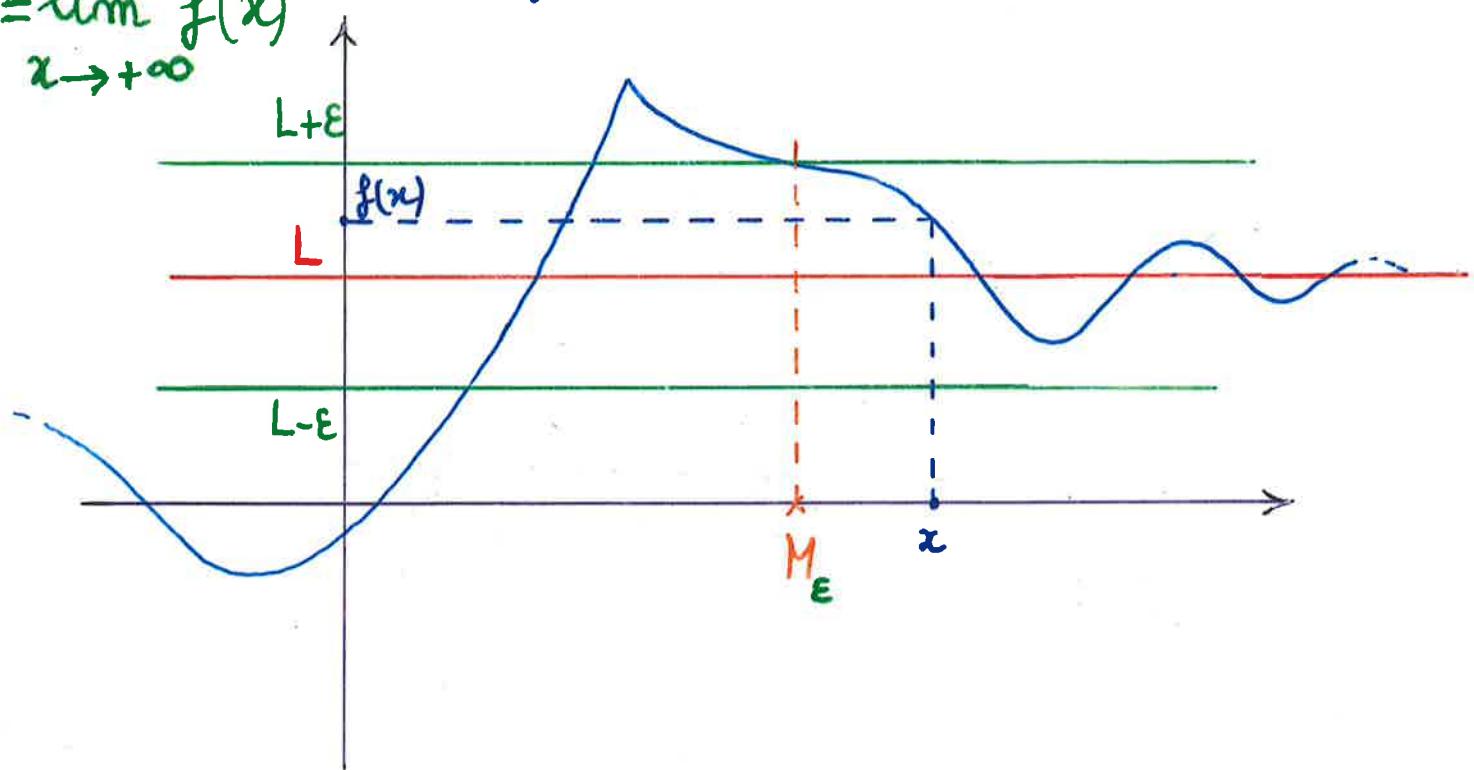
$$\underline{\forall N > 0 \exists M > 0 : \forall x \leq -M f(x) \geq N},$$

$$*\text{ ii) } \lim_{x \rightarrow -\infty} f(x) = -\infty \iff$$

$$\underline{\forall N > 0 \exists M > 0 : \forall x \leq -M f(x) \leq -N}.$$

$$L = \lim_{x \rightarrow +\infty} f(x)$$

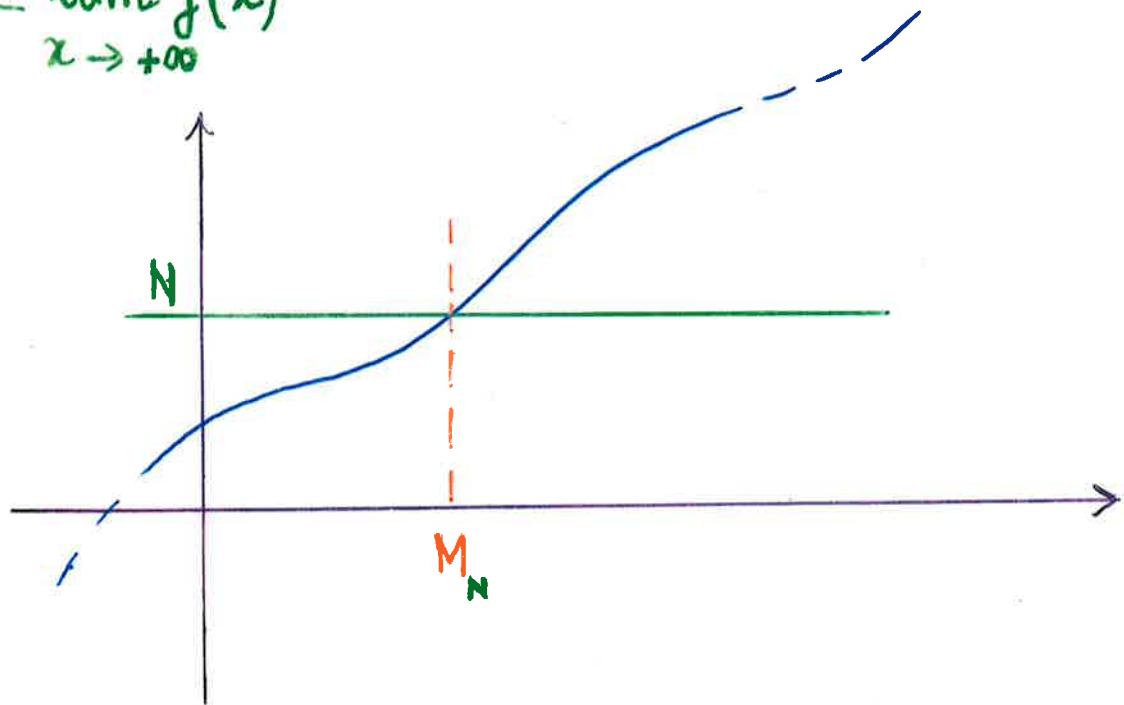
$$f: \mathbb{R} \rightarrow \mathbb{R}$$



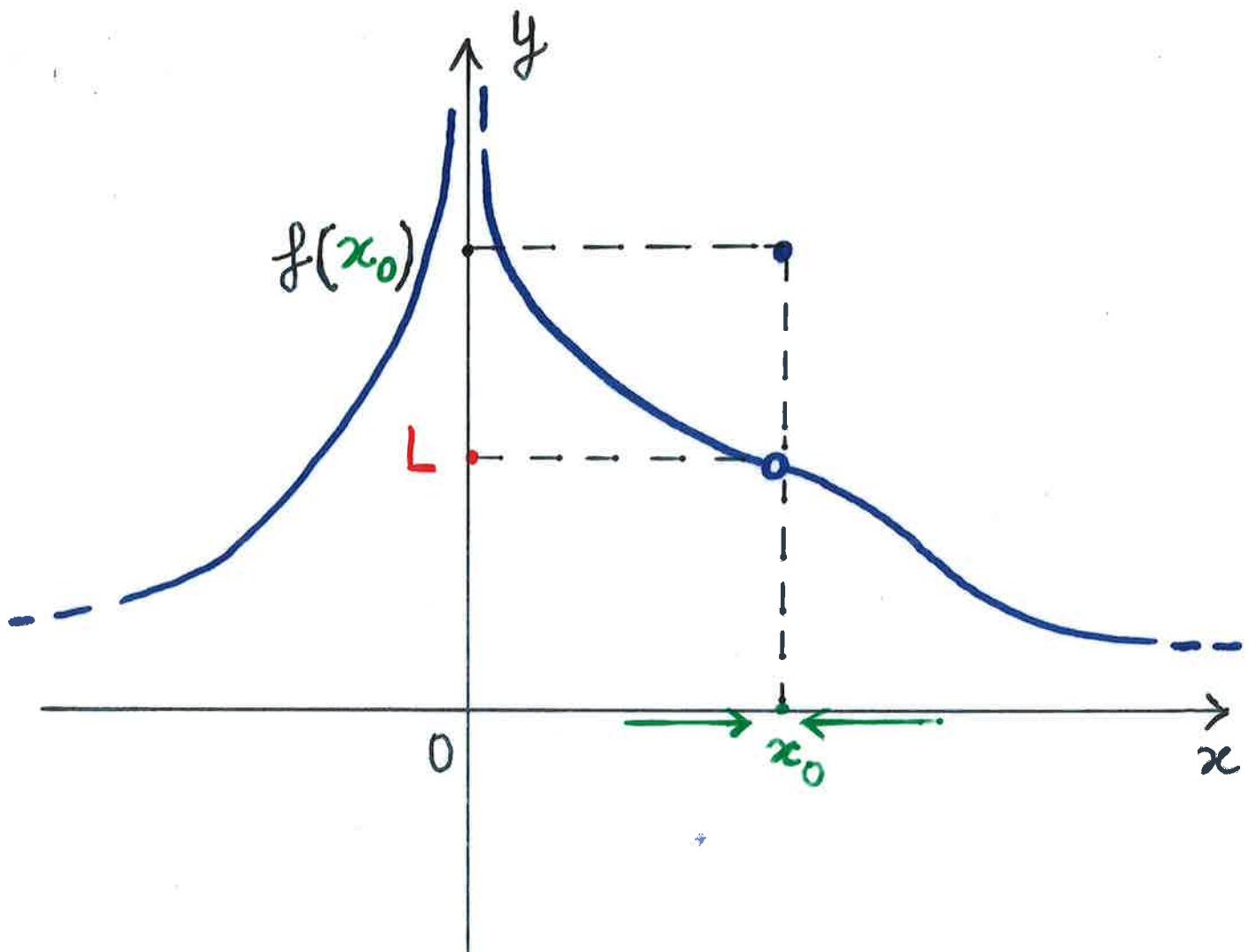
$$\forall \varepsilon > 0 \quad \exists M_\varepsilon > 0 : \forall x \geq M_\varepsilon \Rightarrow |f(x) - L| \leq \varepsilon$$

$$(L - \varepsilon \leq f(x) \leq L + \varepsilon)$$

$$+\infty = \lim_{x \rightarrow +\infty} f(x)$$



$$\forall N > 0 \quad \exists M_N > 0 : \forall x \geq M_N \Rightarrow f(x) \geq N$$



$$\lim_{x \rightarrow x_0} f(x) = L \quad \text{NON DIPENDE DA } f(x_0)$$

## LIMITI AL FINITO

**Def.** Sia  $f : \mathbf{R} \rightarrow \mathbf{R}$  e  $x_0 \in \mathbf{R}$ ; diremo che  $f(x)$  tende al numero  $L \in \mathbf{R}$  per  $x \rightarrow x_0$  se

$$\forall \varepsilon > 0 \exists \delta > 0 : |x - x_0| \leq \delta, x \neq x_0 \implies |L - f(x)| \leq \varepsilon.$$

In tal caso il numero  $L$  si dice il limite di  $f$  per  $x \rightarrow x_0$ , e si scrive  $\lim_{x \rightarrow x_0} f(x) = L$ .

NOTA:  $x_0$  non viene preso in considerazione perché non vogliamo che il valore di  $f$  in  $x_0$  influenzi il limite.

Allo stesso modo si definiscono

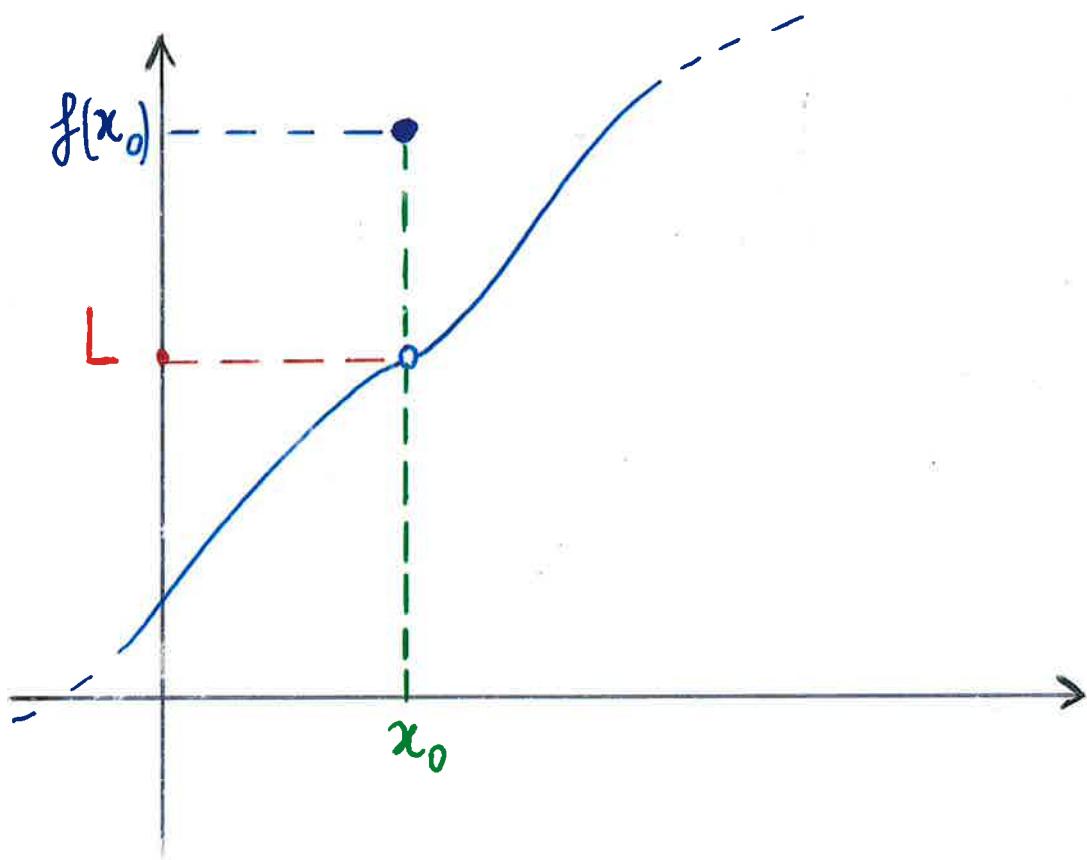
i)  $\lim_{x \rightarrow x_0} f(x) = +\infty \iff$

$$\forall N > 0 \exists \delta > 0 : \forall x : |x - x_0| \leq \delta, x \neq x_0 \implies f(x) \geq N,$$

ii)  $\lim_{x \rightarrow x_0} f(x) = -\infty \iff$

$$\forall N > 0 \exists \delta > 0 : \forall x : |x - x_0| \leq \delta, x \neq x_0 \implies f(x) \leq -N.$$

$$L = \lim_{x \rightarrow x_0} f(x) \quad (f: \mathbb{R} \rightarrow \mathbb{R})$$

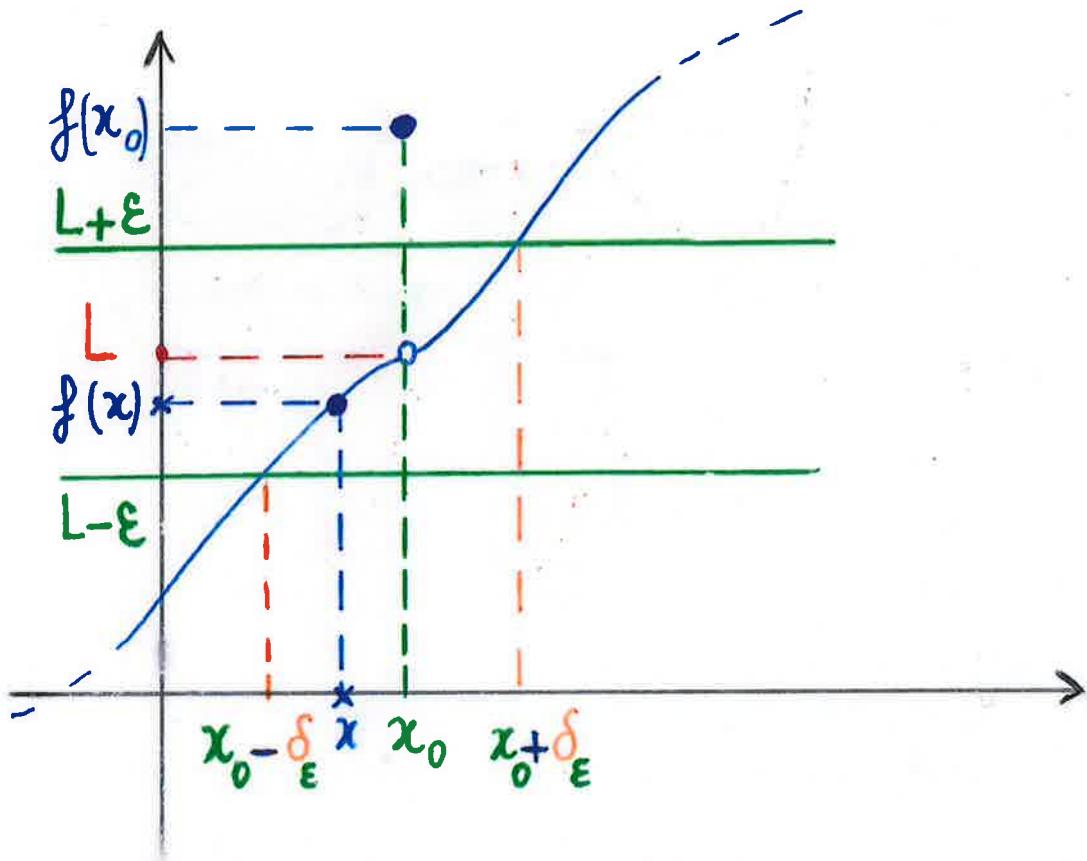


$\forall \varepsilon > 0 \exists \delta_\varepsilon > 0 : \forall x, |x - x_0| \leq \delta_\varepsilon \text{ e } x \neq x_0$   
 $(x_0 - \delta_\varepsilon \leq x \leq x_0 + \delta_\varepsilon)$

$$\Rightarrow |f(x) - L| \leq \varepsilon$$

$$(L - \varepsilon \leq f(x) \leq L + \varepsilon)$$

$$L = \lim_{x \rightarrow x_0} f(x) \quad (f: \mathbb{R} \rightarrow \mathbb{R})$$



$\forall \varepsilon > 0 \exists \delta_\varepsilon > 0 : \forall x, |x - x_0| \leq \delta_\varepsilon \text{ and } x \neq x_0$   
 $(x_0 - \delta_\varepsilon \leq x \leq x_0 + \delta_\varepsilon)$

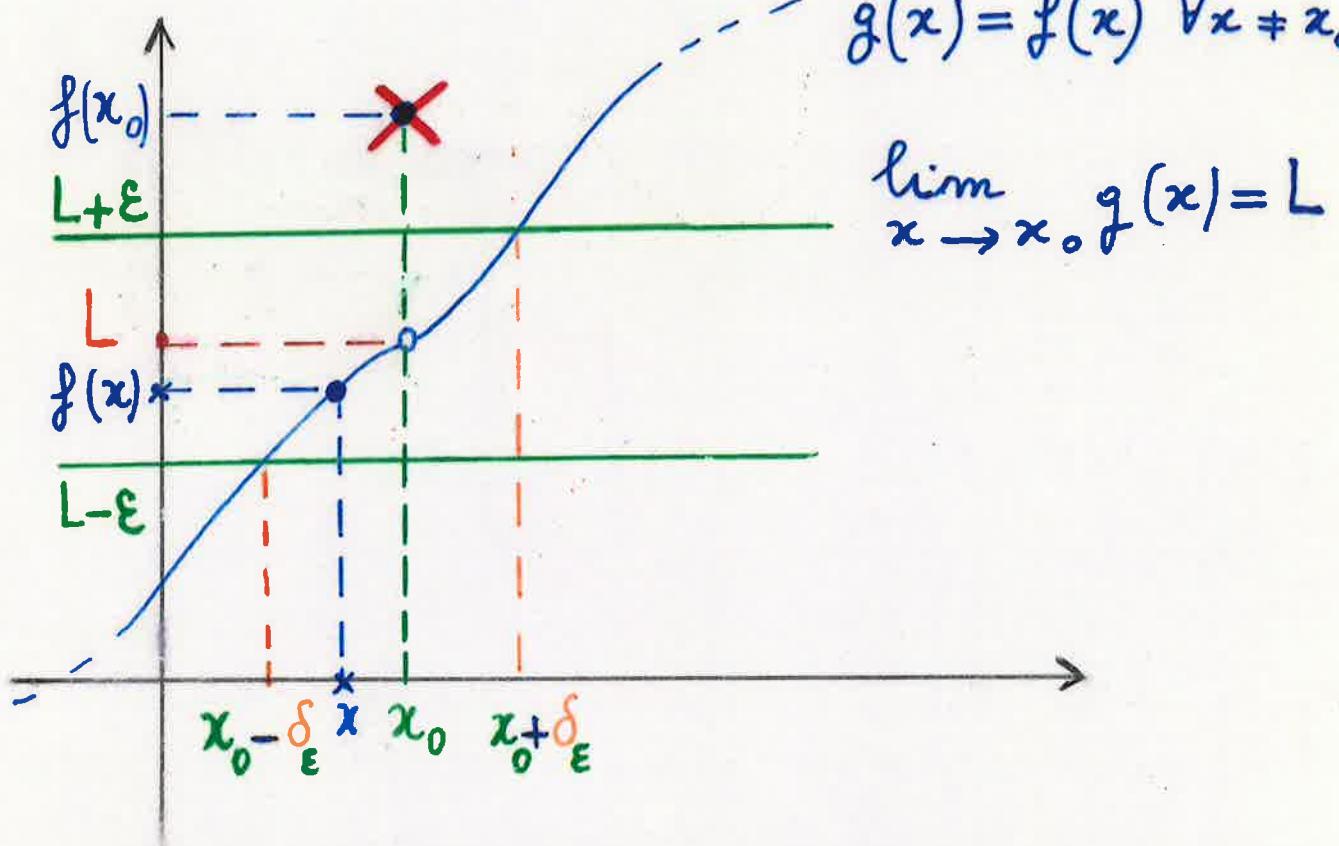
$$\Rightarrow |f(x) - L| \leq \varepsilon$$

$$(L - \varepsilon \leq f(x) \leq L + \varepsilon)$$

$$L = \lim_{x \rightarrow x_0} f(x) \quad (f: \mathbb{R} \rightarrow \mathbb{R})$$

$$g: \mathbb{R} \setminus \{x_0\} \longrightarrow \mathbb{R}$$

$$g(x) = f(x) \quad \forall x \neq x_0$$



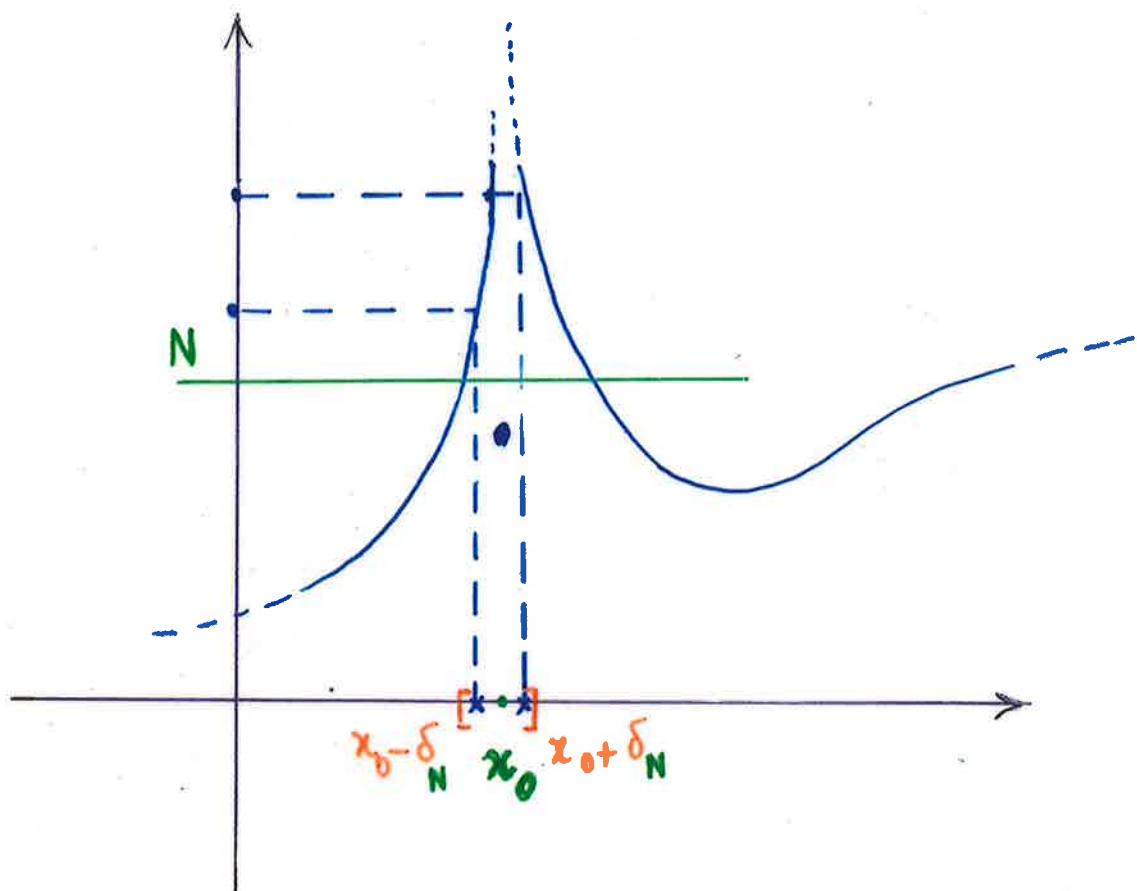
$\forall \varepsilon > 0 \exists \delta_\varepsilon > 0 : \forall x, |x - x_0| \leq \delta_\varepsilon \text{ and } x \neq x_0$

$$(x_0 - \delta_\varepsilon \leq x \leq x_0 + \delta_\varepsilon)$$

$$\Rightarrow |f(x) - L| \leq \varepsilon$$

$$(L - \varepsilon \leq f(x) \leq L + \varepsilon)$$

$$\lim_{x \rightarrow x_0} f(x) = +\infty \quad (f: \mathbb{R} \rightarrow \mathbb{R})$$

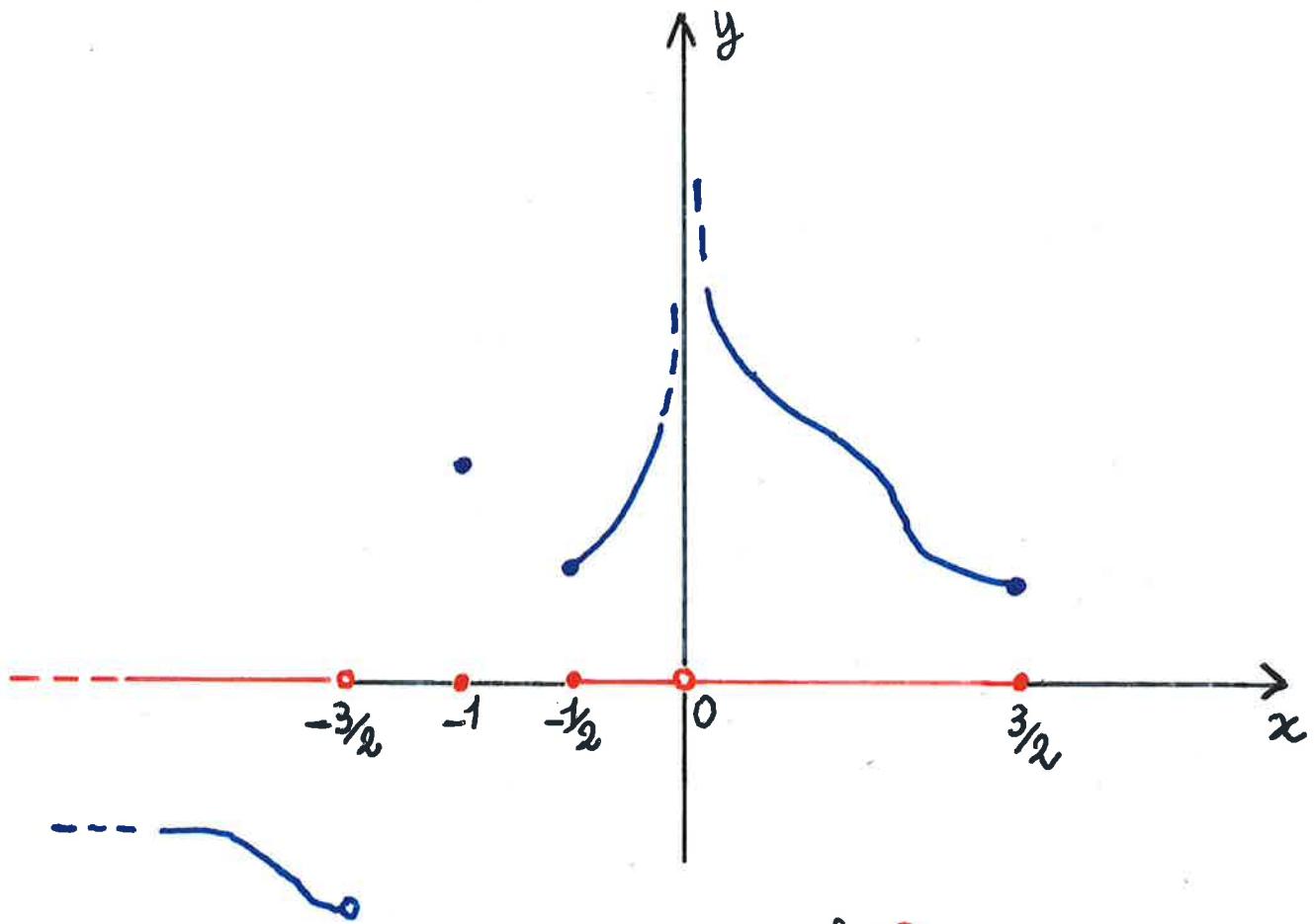


$\forall N > 0 \exists \delta_N > 0 : \forall x, |x - x_0| \leq \delta_N, x \neq x_0,$

$$f(x) \geq N$$

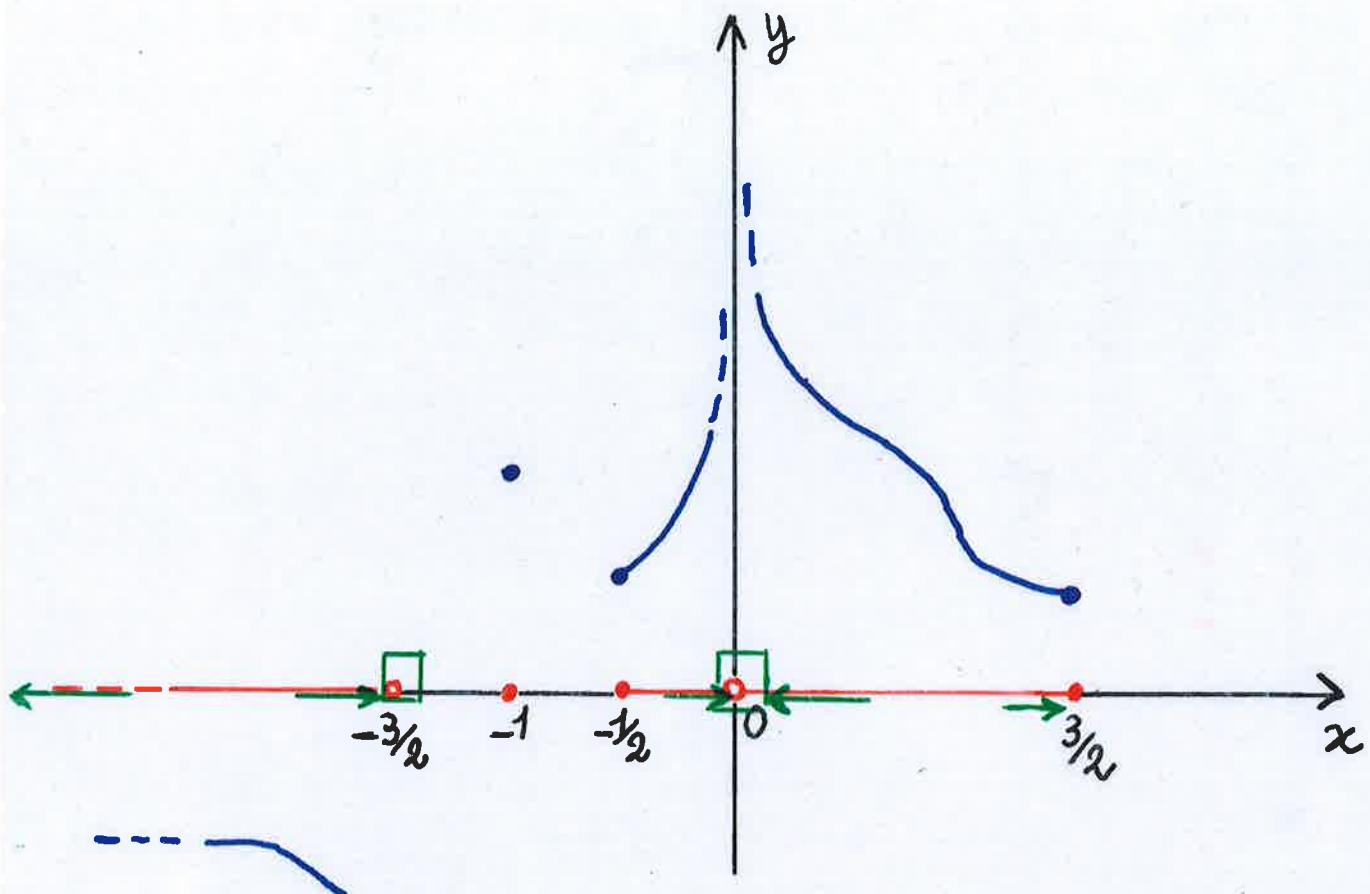
NOTA: le definizioni di limite ora date si possono riasumere con il linguaggio degli intorni: siano  $L, x_0 \in \overline{\mathbf{R}}$ ; allora

$$\lim_{x \rightarrow x_0} f(x) = L \iff \forall \text{ intorno } I \text{ di } L \text{ esiste un intorno } J \text{ di } x_0 \text{ tale che se } x_0 \neq x \in J, \text{ allora } f(x) \in I.$$



$$f: A \rightarrow \mathbb{R}$$

$$A = ]-\infty, -\frac{3}{2}[ \cup \{-1\} \cup [-\frac{1}{2}, 0[ \cup ]0, \frac{3}{2}]$$

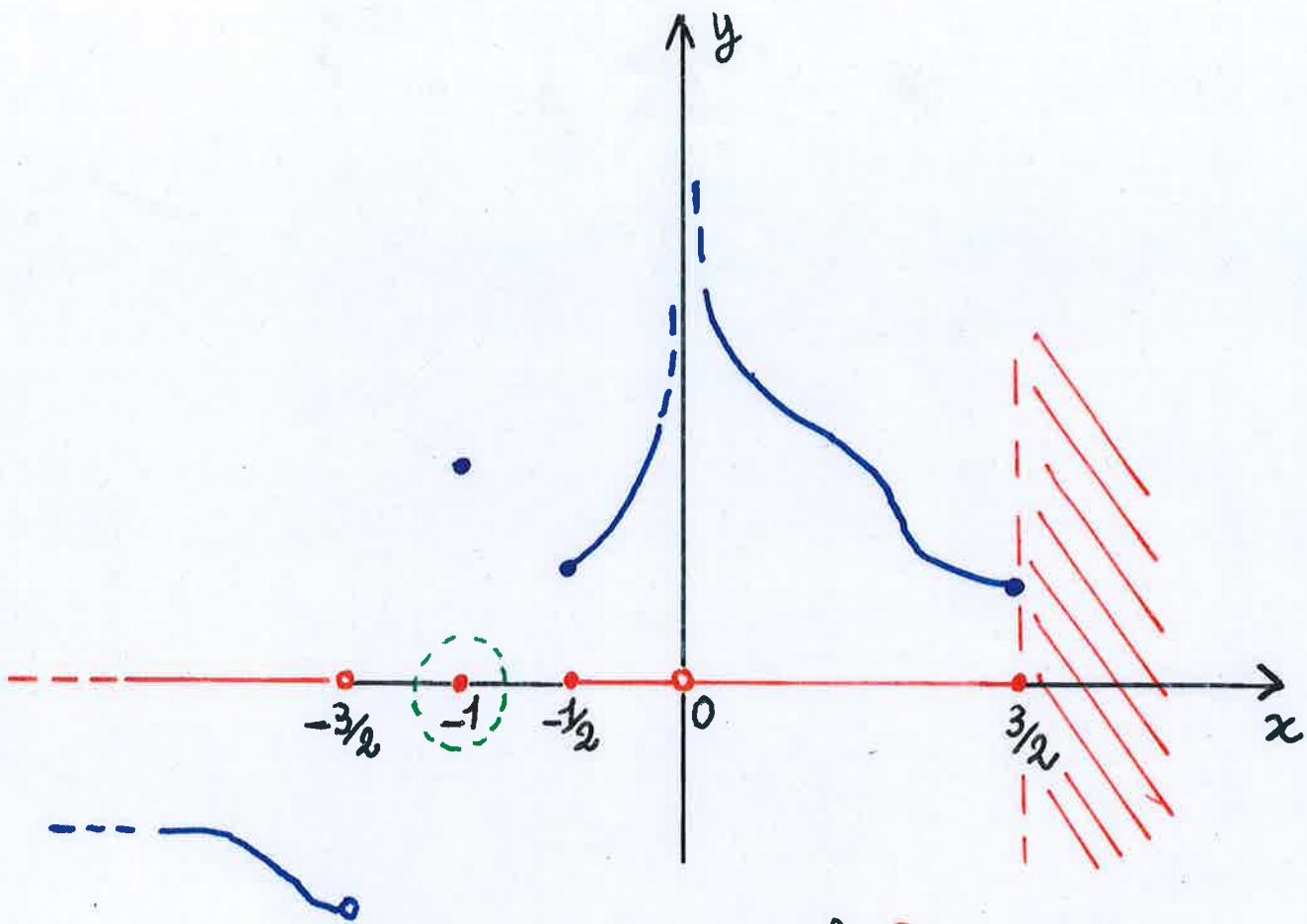


$$f: A \rightarrow \mathbb{R}$$

$$A = ]-\infty, -\frac{3}{2}[ \cup \{-1\} \cup [-\frac{1}{2}, 0[ \cup ]0, \frac{3}{2}]$$

$$\lim_{x \rightarrow -\infty} f(x), \quad \lim_{x \rightarrow -\frac{3}{2}} f(x), \quad \lim_{x \rightarrow 0} f(x)$$

$$\lim_{x \rightarrow \frac{3}{2}} f(x)$$



$$f: A \rightarrow \mathbb{R}$$

$$A = ]-\infty, -\frac{3}{2}[ \cup \{-1\} \cup [-\frac{1}{2}, 0[ \cup ]0, \frac{3}{2}]$$

$$\lim_{\substack{x \rightarrow +\infty}} f(x) \quad \text{NO}$$

$$\lim_{\substack{x \rightarrow -1}} f(x) \quad \text{NO}$$