Linear stability of contact discontinuities for the nonisentropic Euler equations in two space dimensions

Alessandro Morando, Paola Trebeschi
Department of Mathematics
University of Brescia

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1. The Euler equations

We study the nonisentropic Euler equations of a perfect polytropic ideal gas in the plane $\mathbb{R}^2$:

$$\boxed{\begin{aligned} &\partial_t p + \mathbf{u}\cdot\nabla p + \gamma p \nabla\cdot\mathbf{u} = 0,\\ &\rho(\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u}) + \nabla p = 0,\\ &\partial_t S + \mathbf{u}\cdot\nabla S = 0. \end{aligned}} \tag{1}$$

$x = (x_1, x_2) \in \mathbb{R}^2$,

$p = p(t,x) \in \mathbb{R}$ the pressure,

$\mathbf{u}(t,x) \in \mathbb{R}^2$ the velocity,

$S = S(t,x) \in \mathbb{R}$ the entropy,

ρ is the density, obeying the constitutive law

$$\rho(p,S) = Ap^{\frac{1}{\gamma}} e^{-\frac{S}{\gamma}}$$

with $A > 0$ given, $\gamma > 1$ adiabatic number.

Problem: Stability of "contact discontinuities"

Result: Under a "supersonic" condition (that precludes violent instability) we prove an ENERGY ESTIMATE for the linearized problem.

2. Discontinuities

2.1. Contact discontinuities

Let $\Gamma := \{x_2 = \varphi(t, x_1)\}$ be a smooth hypersurphace and

$$(p, \mathbf{u}, S) := \begin{cases} (p^+, \mathbf{u}^+, S^+) & \text{if } x_2 > \varphi(t, x_1) \\ (p^-, \mathbf{u}^-, S^-) & \text{if } x_2 < \varphi(t, x_1), \end{cases}$$

a smooth function on either side of Γ.

Definition 1. $(p, \mathbf{u}, S)$ *is a contact discontinuity solution of (1) if it is a classical solution of (1) on both sides of* Γ *and satisfies the Rankine-Hugoniot jump conditions at* Γ*:*

$$\begin{aligned} \partial_t \varphi = \mathbf{u}^+ \cdot \nu = \mathbf{u}^- \cdot \nu \,, \\ p^+ = p^- \,, \end{aligned}$$

where

$\nu := (-\partial_{x_1}\varphi, 1)$ is a (space) normal vector to Γ.

These conditions yield that

- the normal velocity and pressure are continuous across the interface Γ,
- the *only jumps* experimented by the solution concern the *tangential velocity* and the entropy.

Thus, a contact discontinuity is a *vortex sheet*.

2.2. Planar contact discontinuities

A *planar contact discontinuity* is a piecewise constant solution to (1)

$$(p, \mathbf{u}, S) = \begin{cases} (p_r, \mathbf{u}_r, S_r), & \text{if } x_2 > \sigma t + n x_1, \\ (p_l, \mathbf{u}_l, S_l), & \text{if } x_2 < \sigma t + n x_1. \end{cases}$$

$\mathbf{u}_{r,l} = (v_{r,l}, u_{r,l})^T$ are fixed vectors in $\mathbb{R}^2$, $p_{r,l} > 0, S_{r,l}, \sigma, n$ are fixed real numbers.

• The previous quantities are related by the Rankine-Hugoniot jump conditions

$$\boxed{\begin{aligned} &\sigma + v_r n - u_r = 0, \\ &\sigma + v_l n - u_l = 0, \\ &p_r = p_l =: p. \end{aligned}}$$

• Without loss of generality, we may assume

$$\boxed{\sigma = u_r = u_l = 0\,, \quad n = 0\,, \quad v_r + v_l = 0 \quad (v_r \neq 0).}$$

This corresponds to the following

$$U_{r,l} = (p, v_{r,l}, 0, S_{r,l})^T, \quad \text{with} \quad v_r + v_l = 0.$$

2.3. Stability of nonisentropic planar contact discontinuities in dimension $d = 2, 3$

(Fejer-Miles, 1958-1963 and Coulombel-Morando 2004)

Consider a planar contact discontinuity $U_{r,l} = (p, v_{r,l}, 0, S_{r,l})^T$ with $v_r + v_l = 0$ ($v_{r,l}$ are the tangential velocities) and linearize the Euler equations and the jump conditions around this solution.

- if $d = 3$, the linearized equations do not satisfy the Lopatinskii condition $\Rightarrow$ *violent instability*;

- if $d = 2$ and $\frac{|v_r - v_l|}{2} < \frac{1}{2}\left(c_r^{2/3} + c_l^{2/3}\right)^{3/2}$ the linearized equations do not satisfy the Lopatinskii condition $\Rightarrow$ *violent instability*;

- if $d = 2$ and $\frac{|v_r - v_l|}{2} > \frac{1}{2}\left(c_r^{2/3} + c_l^{2/3}\right)^{3/2}$ the linearized equations satisfy the weak Lopatinskii condition $\Rightarrow$ *weak stability*,

$c(\rho) := \sqrt{p'(\rho)}$ is the sound speed and c_r, c_l are the constant values of $c(\rho)$ in both sides of Γ.

• In any case, the uniform Kreiss-Lopatinskii condition <u>is never satisfied</u>.

3. Energy estimates in two space dimensions

3.1. Reformulation of the problem in a fixed domain

The interface $\Gamma := \{x_2 = \varphi(t, x_1)\}$ is unknown so that the problem is a *free boundary problem*.

• In order to work in a fixed domain we introduce the change of variables

$$\begin{aligned}
&(\tau, y_1, y_2) \to (t, x_1, x_2),\\
&(t, x_1) = (\tau, y_1),\\
&x_2 = \Phi(\tau, y_1, y_2),
\end{aligned}$$

where

$$\begin{aligned}
&\Phi : \mathbb{R}^3 \to \mathbb{R},\\
&\Phi(\tau, y_1, 0) = \varphi(\tau, y_1), \qquad \partial_{y_2}\Phi(\tau, y_1, y_2) \geq \kappa > 0.
\end{aligned}$$

• We define the new unknowns

$$\begin{aligned}
&(p_\sharp^+, \mathbf{u}_\sharp^+, S_\sharp^+)(\tau, y_1, y_2) := (p, \mathbf{u}, S)(\tau, y_1, \Phi(\tau, y_1, y_2))\,,\\
&(p_\sharp^-, \mathbf{u}_\sharp^-, S_\sharp^-)(\tau, y_1, y_2) := (p, \mathbf{u}, S)(\tau, y_1, \Phi(t, y_1, -y_2))\,.
\end{aligned}$$

The functions $p_\sharp^\pm, \mathbf{u}_\sharp^\pm, S_\sharp^\pm$ are smooth on the fixed domain $\{y_2 > 0\}$.

• For convenience, we drop the $\sharp$ index and only keep the $+$ and $-$ exponents.

• Finally, we write (t, x_1, x_2) instead of (τ, y_1, y_2).

• Let us set $\mathbf{u} = (v, u)$. The existence of compressible vortex sheets amounts to prove the existence of smooth solutions of the first order system

$$\boxed{\begin{aligned}
&\partial_t p^+ + v^+\partial_{x_1} p^+ + \left(u^+ - \partial_t\Phi^+ - v^+\partial_{x_1}\Phi^+\right)\frac{\partial_{x_2}p^+}{\partial_{x_2}\Phi^+}\\
&+\gamma p^+\partial_{x_1}v^+ + \gamma p^+\frac{\partial_{x_2}u^+}{\partial_{x_2}\Phi^+} - \gamma p^+\frac{\partial_{x_1}\Phi^+}{\partial_{x_2}\Phi^+}\partial_{x_2}v^+ = 0,\\
&\partial_t v^+ + v^+\partial_{x_1} v^+ + \left(u^+ - \partial_t\Phi^+ - v^+\partial_{x_1}\Phi^+\right)\frac{\partial_{x_2}v^+}{\partial_{x_2}\Phi^+}\\
&+\frac{1}{\rho^+}\partial_{x_1}p^+ - \frac{1}{\rho^+}\frac{\partial_{x_1}\Phi^+}{\partial_{x_2}\Phi^+}\partial_{x_2}p^+ = 0,\\
&\partial_t u^+ + v^+\partial_{x_1} u^+ + \left(u^+ - \partial_t\Phi^+ - v^+\partial_{x_1}\Phi^+\right)\frac{\partial_{x_2}u^+}{\partial_{x_2}\Phi^+}\\
&+\frac{1}{\rho^+}\frac{\partial_{x_2}p^+}{\partial_{x_2}\Phi^+} = 0,\\
&\partial_t S^+ + v^+\partial_{x_1} S^+ + \left(u^+ - \partial_t\Phi^+ - v^+\partial_{x_1}\Phi^+\right)\frac{\partial_{x_2}S^+}{\partial_{x_2}\Phi^+} = 0,
\end{aligned}}$$

in the fixed domain $\{x_2 > 0\}$, where

$$\Phi^{\pm}(t, x_1, x_2) := \Phi(t, x_1, \pm x_2)$$

$((p^-, v^-, u^-, S^-, \Phi^-)$ must solve a similar system)

fulfilling the boundary conditions

$$\boxed{\begin{aligned}
&\Phi^+_{|x_2=0} = \Phi^-_{|x_2=0} = \varphi,\\
&\partial_t\varphi = -v^+_{|x_2=0}\partial_{x_1}\varphi + u^+_{|x_2=0} = -v^-_{|x_2=0}\partial_{x_1}\varphi + u^-_{|x_2=0},\\
&p^+_{|x_2=0} = p^-_{|x_2=0}.
\end{aligned}}$$

The functions $\Phi^{\pm}$ should also satisfy

$$\partial_{x_2}\Phi^+(t, x_1, x_2) \geq k,\ \ \partial_{x_2}\Phi^-(t, x_1, x_2) \leq -k.$$

• The equations are not sufficient to determine the unknowns $U^\pm := (p^\pm, v^\pm, u^\pm, S^\pm)$ and $\Phi^\pm$: another equation relating Φ, U and φ is needed in order to close the system.

We may prescribe that $\Phi^\pm$ solve in the domain $\{x_2 > 0\}$ the *eikonal* equations

$$\boxed{\partial_t \Phi^\pm + v^\pm \partial_{x_1} \Phi^\pm - u^\pm = 0\,.}$$

With this choice the boundary matrix of the system for $U^\pm$ has constant rank in the whole domain $\{x_2 \geq 0\}$ and not only on the boundary $\{x_2 = 0\}$.

• *Matrix form of the system*

For all $U = (p, v, u, S)^T$, we get the system (in the interior $\{x_2 > 0\}$)

$$\boxed{\begin{array}{l} L(U^+, \nabla\Phi^+)U^+ = 0, \\ L(U^-, \nabla\Phi^-)U^- = 0, \\ \partial_t \Phi^\pm + v^\pm \partial_{x_1} \Phi^\pm - u^\pm = 0 \end{array}}$$

where

$$\begin{aligned} L(U^+, \nabla\Phi^+)U^+ = \partial_t U^+ + A_1(U^+)\partial_{x_1} U^+ + \\ + \tfrac{1}{\partial_{x_2}\Phi^+}\left(A_2(U^+) - \partial_t \Phi^+ I_4 - \partial_{x_1}\Phi^+ A_1(U^+)\right)\partial_{x_2} U^+ \end{aligned}$$

with the boundary conditions (on $\{x_2 = 0\}$)

$$\boxed{\begin{array}{r} \Phi^+_{|_{x_2=0}} = \Phi^-_{|_{x_2=0}} = \varphi\,, \\ \partial_t \varphi = -v^+_{|_{x_2=0}}\,\partial_{x_1}\varphi + u^+_{|_{x_2=0}} = -v^-_{|_{x_2=0}}\,\partial_{x_1}\varphi + u^-_{|_{x_2=0}} \\ p^+_{|_{x_2=0}} = p^-_{|_{x_2=0}}. \end{array}}$$

3.2. The linearized equations

• Consider a planar contact discontinuity

$$U_r = \begin{pmatrix} p \\ v_r \\ 0 \\ S_r \end{pmatrix}, \quad U_l = \begin{pmatrix} p \\ v_l \\ 0 \\ S_l \end{pmatrix}, \quad \Phi_{r,l} = \pm x_2.$$

We assume that $v_r + v_l = 0$, $v_r > 0$.

• Let us consider

$$U_{r,l} + \varepsilon \dot{U}_\pm, \quad \pm x_2 + \varepsilon \dot{\psi}_\pm$$

where we have set $\dot{U}_\pm = (\dot{p}_\pm, \dot{\mathbf{u}}_\pm, \dot{S}_\pm)$, $\dot{\psi}_\pm$ is a small perturbation of the exact solution $U_{r,l}, \Phi_{r,l}$.

• Up to second order, $\dot{U}_\pm = (\dot{p}_\pm, \dot{\mathbf{u}}_\pm, \dot{S}_\pm)^T$ must solve the system

$$\boxed{\begin{array}{l} L'\dot{U} = 0, \text{ in } \{x_2 > 0\}, \\ B(\dot{U}, \psi) = 0, \text{ on } \{x_2 = 0\}, \end{array}}$$

where we have set for shortness $\dot{U} := (\dot{U}_+, \dot{U}_-)^T$,

$$L'\dot{U} := \partial_t \dot{U} + \begin{pmatrix} A_1(U_r) & 0 \\ 0 & A_1(U_l) \end{pmatrix} \partial_{x_1} \dot{U} + \begin{pmatrix} A_2(U_r) & 0 \\ 0 & -A_2(U_l) \end{pmatrix} \partial_{x_2} \dot{U},$$

and

$$B(\dot{U}, \psi) := \begin{pmatrix} (v_r - v_l)\partial_{x_1}\psi - (\dot{u}_+ - \dot{u}_-) \\ \partial_t \psi + v_r \partial_{x_1}\psi - \dot{u}_+ \\ p_+ - p_- \end{pmatrix}.$$

• We have to introduce source terms; we study

$$\boxed{\begin{array}{l} L'\dot{U} = f \text{ in } \{x_2 > 0\}, \\ B(\dot{U}, \psi) = g, \text{ on } \{x_2 = 0\} \end{array}}$$

• It is useful make another linear change of unknowns

$$W_1 := \dot{v}_+,\ W_2 := \frac{1}{2}\left(-\frac{\dot{p}_+}{\gamma p} + \frac{\dot{u}_+}{c_r}\right)\ W_3 := \frac{1}{2}\left(\frac{\dot{p}_+}{\gamma p} + \frac{\dot{u}_+}{c_r}\right),\ W_4 := \dot{S}_+,$$
$$W_5 := \dot{v}_-,\ W_6 := \frac{1}{2}\left(-\frac{\dot{p}_-}{\gamma p} + \frac{\dot{u}_-}{c_l}\right)\ W_7 := \frac{1}{2}\left(\frac{\dot{p}_-}{\gamma p} + \frac{\dot{u}_-}{c_l}\right),\ W_8 := \dot{S}_-$$

• This change of unknowns W transform the system $L'\dot{U} = f$ in a symmetric hyperbolic form

$$\boxed{\begin{array}{l} \mathcal{L}W = f, \text{ in } \{x_2 > 0\} \\ \mathcal{B}(W^{\mathrm{nc}}, \psi) = g, \text{ on } \{x_2 = 0\}, \end{array}}$$

with new data f, g, and

$$\mathcal{L}W := \mathcal{A}_0\partial_t W + \mathcal{A}_1\partial_{x_1} W + \mathcal{A}_2\partial_{x_2} W$$
$$\mathcal{B}(W^{\mathrm{nc}}, \psi) := \underline{M}W^{\mathrm{nc}} + \underline{b}\begin{pmatrix} \partial_t\psi \\ \partial_{x_1}\psi \end{pmatrix}.$$

$W := (W_1, W_2, W_3, W_4, W_5, W_6, W_7, W_8)^T$,
$W^{\mathrm{c}} := (W_1, W_4, W_5, W_8)^T$,
$W^{\mathrm{nc}} := (W_2, W_3, W_6, W_7)^T =$ (linear combination of p, u)

$$\underline{M} := \begin{pmatrix} -c_r & -c_r & c_l & c_l \\ -c_r & -c_r & 0 & 0 \\ -1 & 1 & 1 & -1 \end{pmatrix}, \qquad \underline{b} := \begin{pmatrix} 0 & v_r - v_l \\ 1 & v_r \\ 0 & 0 \end{pmatrix}.$$

• We want to find an L^2 a priori estimate of the solution to the linearized problem

$$\boxed{\begin{array}{l} \mathcal{L}W = f, \text{ in } \{x_2 > 0\} \\ \mathcal{B}(W^{\mathrm{nc}}, \psi) = g, \text{ on } \{x_2 = 0\}, \end{array}}$$

in

$$\Omega := \{(t, x_1, x_2) \in \mathbb{R}^3 \text{ s.t. } x_2 > 0\} = \mathbb{R}^2 \times \mathbb{R}^+ .$$

The boundary $\partial\Omega = \{x_2 = 0\}$ is identified to $\mathbb{R}^2$.

3.3. The functional setting

• Define

$$H^s_\gamma(\mathbb{R}^2) := \{u \in \mathcal{D}'(\mathbb{R}^2) \text{ s.t. } \exp(-\gamma t)u \in H^s(\mathbb{R}^2)\} ,$$

equipped with the norm

$$\|u\|_{H^s_\gamma(\mathbb{R}^2)} := \|\exp(-\gamma t)u\|_{H^s(\mathbb{R}^2)} .$$

• Define similarly the space $H^s_\gamma(\Omega)$.

• The space $L^2(\mathbb{R}^+; H^s_\gamma(\mathbb{R}^2)$ is the space of all functions $v = v(t, x_1, x_2)$ in Ω such that the following norm

$$|||v|||^2_{L^2(H^s_\gamma)} := \int_0^{+\infty} \|v(\cdot, x_2)\|^2_{H^s_\gamma(\mathbb{R}^2)} dx_2$$

is finite.

3.4. The main result.

Theorem 1. *Let $(U_{r,l}, \Phi_{r,l})$ be a planar contact discontinuity (unperturbed solution).*

i) If $v_r - v_l > \left(c_r^{\frac{2}{3}} + c_l^{\frac{2}{3}}\right)^{\frac{3}{2}}$ and $v_r - v_l \neq \sqrt{2}\,(c_r + c_l)$ *then there exists $C > 0$ such that for all $\gamma \geq 1$ and all $(W, \psi) \in H^2_\gamma(\Omega) \times H^2_\gamma(\mathbb{R}^2)$*

$$\begin{aligned}&\gamma |||W|||^2_{L^2_\gamma(\Omega)} + \|W^{\mathsf{nc}}_{|x_2=0}\|^2_{L^2_\gamma(\mathbb{R}^2)} + \|\psi\|^2_{H^1_\gamma(\mathbb{R}^2)}\\ &\leq C\left(\tfrac{1}{\gamma^3}|||\mathcal{L}W|||^2_{L^2(H^1_\gamma)} + \tfrac{1}{\gamma^2}\|\mathcal{B}(W^{\mathsf{nc}}, \psi)\|^2_{H^1_\gamma(\mathbb{R}^2)}\right).\end{aligned}$$

ii) If $v_r - v_l = \sqrt{2}\,(c_r + c_l)$ then there exists $C > 0$ such that for all $\gamma \geq 1$ and all $(W, \psi) \in H^3_\gamma(\Omega) \times H^3_\gamma(\mathbb{R}^2)$

$$\begin{aligned}&\gamma |||W|||^2_{L^2_\gamma(\Omega)} + \|W^{\mathsf{nc}}_{|x_2=0}\|^2_{L^2_\gamma(\mathbb{R}^2)} + \|\psi\|^2_{H^1_\gamma(\mathbb{R}^2)}\\ &\leq C\left(\tfrac{1}{\gamma^5}|||\mathcal{L}W|||^2_{L^2(H^2_\gamma)} + \tfrac{1}{\gamma^4}\|\mathcal{B}(W^{\mathsf{nc}}, \psi)\|^2_{H^2_\gamma(\mathbb{R}^2)}\right).\end{aligned}$$

- If *i)*: loss of **one** derivative for W; no loss of derivatives for the front function ψ;

- If *ii)*: loss of **two** derivatives for W; loss of **one** derivative for the front function ψ;

- Only the <u>trace of the non-characteristic part</u> of the solution can be controlled at the boundary since the problem is characteristic. The loss of control regards the tangential velocity and the entropy.

Further comments

- If we are in the "isentropic case"

$$c_r = c_l := c$$

  both values $(c_r^{\frac{2}{3}} + c_l^{\frac{2}{3}})^{\frac{3}{2}}$ and $\sqrt{2}(c_r + c_l)$ entailed in *i)*,*ii)* actually are reduced to $2\sqrt{2}c$. In this case, the assumption $v_r - v_l > 2\sqrt{2}c$ in *i)* prevents the occurrence of *ii)*;

- If we are in in the "full nonisentropic" situation $c_r \neq c_l$ the value $\sqrt{2}\,(c_r + c_l)$ is <u>strictly greater</u> than $(c_r^{\frac{2}{3}} + c_l^{\frac{2}{3}})^{\frac{3}{2}}$ so that <u>*ii)* has to be accounted</u>.

4. Main steps of the proof

4.1. Reformulating the problem

Step 1. It is convenient rewriting the system in terms of the new unknown $\widetilde{W} := \exp(-\gamma t)W$ and $\widetilde{\psi} := \exp(-\gamma t)\psi$.

Then we consider the following system

$$\boxed{\begin{aligned} &\mathcal{L}^\gamma \widetilde{W} := \gamma \mathcal{A}_0 \widetilde{W} + \mathcal{L}\widetilde{W} = \exp(-\gamma t) f, \\ &\mathcal{B}^\gamma(\widetilde{W}^{\mathrm{nc}}, \widetilde{\psi}) := \underline{M}\widetilde{W}^{\mathrm{nc}}_{|x_2=0} + \underline{b}\begin{pmatrix} \gamma\widetilde{\psi} + \partial_t\widetilde{\psi} \\ \partial_{x_1}\widetilde{\psi} \end{pmatrix} = \exp(-\gamma t) g. \end{aligned}}$$

Step 2. It is enough to find an energy estimate for the problem (we drop "tilde")

$$\boxed{\begin{aligned} &\mathcal{L}^\gamma W := 0, \\ &\mathcal{B}^\gamma(W^{\mathrm{nc}}, \psi) := G \end{aligned}} \qquad (2)$$

where $G = G(W^{nc}, \psi, f)$.

Since $\mathcal{A}_j$ are symmetric and $\mathcal{A}_0$ is positive definite, taking the scalar product of $(2)_1$ with W we get

$$\gamma |||W|||_0^2 \leq C ||W^{nc}_{|_{x_2=0}}||_0^2.$$

• Hence it remains to find an estimate for $W^{nc}_{|_{x_2=0}}$ and the front ψ.

4.2. Eliminating the front ψ

Step 3. Let W, ψ satisfy

$$\begin{cases} \mathcal{L}^\gamma W := 0, \\ \mathcal{B}^\gamma(W^{\mathsf{nc}}, \psi) := G. \end{cases}$$

• Perform a Fourier transform in (t, x_1). Let us denote the dual variables by (δ, η) and write also $\tau := \delta + i\gamma$.

• The Fourier transform $(\widehat{W}, \widehat{\psi})$ of (W, ψ) must solve the following system of algebraic-differential equations

$$(\tau \mathcal{A}_0 + i\eta \mathcal{A}_1)\widehat{W} + \mathcal{A}_2 \frac{d\widehat{W}}{dx_2} = 0, \quad \text{if } x_2 > 0 \tag{3}$$

$$b(\tau, \eta)\widehat{\psi} + \underline{M}\widehat{W}^{\mathsf{nc}}(0) = \widehat{G}, \tag{4}$$

with $b(\tau, \eta)$ defined by

$$b(\tau, \eta) := \underline{b} \begin{pmatrix} \tau \\ i\eta \end{pmatrix} = \begin{pmatrix} 2i\eta v_r \\ \tau + i\eta v_r \\ 0 \end{pmatrix}.$$

• It can be shown that $b(\tau, \eta)$ satisfies an *elliptic bound*, then we find

$$\|\psi\|_{1,\gamma}^2 \leq C \left(\|W^{\mathsf{nc}}_{|x_2=0}\|_0^2 + \|G\|_0^2 \right)$$

• As a conseguence of the ellipticity of b the boundary condition (4) can be written as

$$\beta(\tau, \eta)\widehat{W}^{\mathsf{nc}}(0) = \widehat{h}$$

where the front ψ disappears.

• We focus on the following problem

$$\boxed{\begin{aligned}&(\tau\mathcal{A}_0+i\eta\mathcal{A}_1)\widehat{W}+\mathcal{A}_2\tfrac{d\widehat{W}}{dx_2}=0, \quad x_2>0,\\ &\beta(\tau,\eta)\widehat{W}^{\mathsf{nc}}(0)=\widehat{h}.\end{aligned}} \tag{5}$$

where the new boundary condition involves only the non characteristic part of $\widehat{W}$.

Step 4. Because of the characteristic boundary some equations in (5) do not entail differentiation with respect to x_2. We are led to the ODE system

$$\boxed{\begin{aligned}&\tfrac{d\widehat{W}^{\mathsf{nc}}}{dx_2}=\mathcal{A}(\tau,\eta)\widehat{W}^{\mathsf{nc}}, \quad x_2>0,\\ &\beta(\tau,\eta)\widehat{W}^{\mathsf{nc}}=\widehat{h}, \quad x_2=0,\end{aligned}} \tag{6}$$

Step 5. We have to prove an energy estimate for the problem (6).

• The technique of the proof is the construction of a degenerate Kreiss symmetrizer.

• Under the supersonic assumptions

$$v_r-v_l>(c_r^{\frac{2}{3}}+c_l^{\frac{2}{3}})^{\frac{3}{2}}$$

the problem satisfies the Kreiss Lopatinskii condition in a weak sense; in other words, the Lopatinskii determinant associated to (6) vanishes at some boundary frequencies (τ,η) with $\Re\tau=0$.

• The failure of the uniform Kreiss-Lopatinskii condition yields a loss of derivatives with respect to the source terms: the loss is strictly related to the order of vanishing of the Lopatinskii determinant.

4.3. Roots of the Lopatinskii determinant

Let $\Delta(\tau,\eta)$ be the Lopatinskii determinant associated to (6). Let

$$\Sigma := \{(\tau,\eta) \in \mathbb{C}\times\mathbb{R} : \ |\tau|^2 + v_r^2\eta^2 = 1,\ \Re\tau \geq 0\}$$

be the *unit hemi-sphere* in the frequency space.

Proposition 1. *Assume that*

$$v_r > \frac{1}{2}(c_r^{\frac{2}{3}} + c_l^{\frac{2}{3}})^{\frac{3}{2}}, \quad v_r + v_l = 0.$$

a. *If $c_r = c_l =: c$ (isentropic case) then there exists a positive number V_1 such that for every $(\tau,\eta) \in \Sigma$, $\Delta(\tau,\eta) = 0$ if and only if*

$$\tau = 0 \quad or \quad \tau = \pm iV_1\eta.$$

Each of the preceding roots of Δ is simple; namely if (τ_0,η_0) is anyone of the points above there exists an open neighborhood $\mathcal{V}$ of (τ_0,η_0) in Σ and a C^∞ function h defined on $\mathcal{V}$ such that for all $(\tau,\eta) \in \mathcal{V}$:

$$\Delta(\tau,\eta) = (\tau-\tau_0)h(\tau,\eta), \quad \text{and} \quad h(\tau_0,\eta_0) \neq 0.$$

b. *If $c_r \neq c_l$ (nonisentropic case) then there exist two positive numbers X_2, X_3 satisfying*

$$c_r - v_r < c_rX_2 < c_rX_3 < -c_l + v_r, \tag{7}$$

such that $\Delta(\tau,\eta)=0$, *for* $(\tau,\eta)\in\Sigma$, *if, and only if,*

$$\tau = iqv_r\eta \quad or \quad \tau = ic_rX_2\eta \quad or \quad \tau = ic_rX_3\eta, \quad (8)$$

where $q := \frac{c_r-c_l}{c_r+c_l}$.

For $v_r \neq \frac{c_r+c_l}{\sqrt{2}}$, *each of the preceding roots of* Δ *is simple and the same situation occurs as in the isentropic case a.*
On the contrary, when $v_r = \frac{c_r+c_l}{\sqrt{2}}$ *one (and only one) of the two identities below holds true*

$$qv_r = c_rX_2 \qquad or \qquad qv_r = c_rX_3.$$

Hence each of the roots $(iqv_r\eta,\eta)\in\Sigma$ *of* Δ *is double. More precisely, this means that to every point* $(iqv_r\eta_0,\eta_0)$ *(with* $\eta_0^2(1-q^2v_r^2)=1$*) there correspond an open neighborhood* $\mathcal{V}$ *in* Σ *and a* C^∞ *function* h *on* $\mathcal{V}$ *such that*

$$\Delta(\tau,\eta) = (\tau - iqv_r\eta_0)^2 h(\tau,\eta), \quad \forall(\tau,\eta)\in\mathcal{V}$$

and $h(iqv_r\eta_0,\eta_0)\neq 0$. *The other root of* Δ *remains simple, as in the case of* $v_r \neq \frac{c_r+c_l}{\sqrt{2}}$.

4.4. Construction of the Kreiss symmetrizer

The construction of the symmetrizer is microlocal and is performed in a neighborhood of different classes of frequency points (τ, η).

a. interior points $(\tau, \eta) : \Re\tau > 0$. Here standard Kreiss symmetrizer exists and L^2-estimate without loss of derivatives is obtained.

b. boundary points $(\tau, \eta) : \Re\tau = 0$. We need to distinguish the following classes:

   b1 Points where $\mathcal{A}(\tau, \eta)$ is diagonalizable and the Kreiss-Lopatinskii condition is satisfied $\Rightarrow$ classical Kreiss techniques apply and give an L^2 estimate with no loss of derivatives.

   b2 Points where $\mathcal{A}(\tau, \eta)$ is diagonalizable and the Lopatinskii condition breaks down (i.e. $\Delta(\tau, \eta) = 0$) $\Rightarrow$ construction of a degenerate Kreiss' symmetrizer: this yields an L^2 estimate with loss of derivatives. The loss of derivatives depends on the multiplicity of the roots of the Lopatinskii determinat Δ:

   - roots of multiplicity 1 give a loss of 1 derivative

   - roots of multiplicity 2 give a loss of 2 derivatives

b3 Points where $\mathcal{A}(\tau,\eta)$ is not diagonalizable. In those points, the Lopatinskii condition is satisfied (i.e. $\Delta(\tau,\eta) \neq 0$). The construction of the simmetrizer follows as in case a. and an L^2-energy estimate without loss of derivatives is obtained.

b4 Poles of $\mathcal{A}(\tau,\eta)$. At those points, the Lopatinskii condition is satisfied. We construct a symmetrizer by working on the original system (5) and the energy estimate is without loss of derivatives

Work in progress

Finding an energy estimate for the variable coefficient problem obtained linearizing the Euler equations around a non constant discontinuity. So far, we have found the energy estimate far from the double roots of the Lopatinskii determinant.

Techniques employed:

- paralinearization

- microlocal analysis

Some bibliography

No general existence theorem for solutions which allows discontinuities.

In the NON CHARACTERISTIC case:

- Complete analysis of existence and stability of a *single shock wave* was made by
  - A. Majda 1983,
  - G. Métivier 2001.
- Existence of *rarefaction waves* by S. Alinhac 1989.
- Existence of *sound waves* by G. Métivier 1991.
- Uniform existence of *shock waves* with small strength by J. Francheteau & G. Métivier 2000.

In the CHARACTERISTIC case (vortex sheet):

- Existence and stability of *vortex sheets* for the *isentropic* Euler equations by Coulombel-Secchi 2004-06,
- stability of *vortex sheets* for the *nonisentropic* Euler equations by Morando-Trebeschi 2006 (in progress).